JOURNAL OF ECONOMIC THEORY 58, 250–289 (1992)

Optimal Fiscal Policy in a Stochastic Growth Model*

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Received December 31, 1990; revised December 10, 1991

1. INTRODUCTION

Although most neoclassical growth models focus on competitive equilibria which are Pareto optimal and ignore the role of government, recent works by Romer [28, 29], Judd [20], Baxter and King [5], and Bizer and Judd [7] have shown that government can be introduced into and suboptimal equilibria can be studied in these models as well. Their studies, however, are positive analyses of the effects of fiscal policy in competitive equilibria. The purpose of this paper is to apply Ramsey's [27] theory of optimal taxation to the normative study of fiscal policy in a stocastic growth model.

Ramsey studied a static, representative consumer economy with many goods. In his formulation, a government consumes a fixed amount of these goods, which it purchases at market prices and finances by levying flat-rate excise taxes, prices and quantities are determined in a competitive equilibrium. The government is benevolent, and its objective is to choose the optimal tax rates to maximize the representative consumer's utility. A number of economists, among them Kydland and Prescott [23], Barro [3], Turnovsky and Brock [31], and Lucas and Stockey [26], have noted that Ramsey's formulation can be applied to dynamic economies with uncertainty if one reinterprets the goods in static problems as dated, state contingent goods. In this framework, a fiscal policy is a rule which specifies state contingent tax rates and the amount of public debt issue in each date. The optimal fiscal policy is defined as the one which leads to a quantity

* This paper is based on my Ph.D. thesis. Financial support from the University of Chicago and the Bradley Foundation is gratefully acknowledged. I thank Andrew Atkeson, David S. Bizer, Phillip A. Braun, Marvin Goodfriend, Milton Harris, Michael Woodford, Lawrence Wu, Chi-Wa Yuen, two anonymous referees and workshop participants at the University of Chicago and Federal Reserve Bank of Minneapolis for their helpful comments and suggestions. Special thanks are due to Robert E. Lucas, Jr. and Kevin M. Murphy for their support and guidance. All the remaining errors are my own.

0022-0531/92 $5.00

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allocation that maximizes the representative consumer's utility, and, at the same time, is consistent with given government consumption and with market determination of prices and quantities. The quantity allocation that achieves maximal consumer utility is called the *Ramsey allocation*, and the corresponding competitive equilibrium is called the *Ramsey equilibrium*.

Using Ramsey's formulation, Lucas and Stokey [26] analyze optimal fiscal policy in a stochastic economy without capital, and Chamley [9] analyzes optimal capital and labor income taxation in a deterministic economy with capital accumulation. This paper extends their analyses of Ramsey taxation to an economy with both capital accumulation and uncertainty. Three issues are addressed in this paper: (1) How to characterize the allocations that are implementable in competitive equilibria with labor and capital income taxes? For any given implementable quantity allocation are there many different fiscal policies that can implement it? (2) What is the structure of Ramsey taxation? That is, what is the structure of optimal capital and labor income taxation if government expenditures play no specific roles in the economy and the government takes their level as exogenously given? (3) What is the dynamics of a Ramsey equilibrium?

In the past decade, there has been a considerable interest in studying optimal fiscal policy in dynamic economies (recent examples of work in this area include Judd [21], Lucas [25], Barro and Sala-i-Martin [4], Chari *et al.* [10], Jones *et al.* [19], King [22], and Yuen [32]) and the three issues this paper addresses have been discussed by many writers in different dynamic models. This paper, however, addresses all three issues in a unified general equilibrium model with both capital accumulation and uncertainty. It demonstrates that the implication of optimal fiscal policy in this model can be very different from the implications that have been drawn from studies in models that ignore some aspects of the model that are considered here.

(1) *Indeterminacy of Fiscal Policy.* Barro [3] studies how the optimal level of public debt payment is determined in a deterministic model where the economy's capital accumulation is exogenously determined independent of tax rates. He shows that, for any given initial level of public debt outstanding, the optimal levels of public debt payment in future periods are determined by Ramsey's theory of optimal taxation. His result is later confirmed by Lucas and Stockey [26] in a stochastic economy without capital. They also show that in the economy with uncertainty, government can achieve a more efficient quantity allocation when it issues state contingent bonds than it could if it issued risk-free bonds. This is because state contingent bonds can help the government to spread tax burden across states as well as over time, and, consequently, can help to reduce distortion. Both of these studies, however, assume that there exists no capital market. In

this paper, by allowing the existence of a capital market along with the government bond market I show that, for any given initial level of public debt outstanding, the optimal levels of public debt payment in future periods are not determinate, neither are the actual optimal capital income tax rates. What is determinate is market values of public debt issue and *ex ante* expected capital income tax rates. I also show that it is not necessary for a government to issue state contingent bonds in order to achieve optimal quantity allocations. Rather, it is demonstrated that government can use state contingent capital income tax to replace state contingent bonds. (See Chari *et al.* [10] for similar results.) Although in this paper the indeterminacy of fiscal policy is analyzed in a representative agent economy, in [33] I show that the indeterminacy of fiscal policy exists in economies with heterogeneous agents as well if one assumes that complete capital markets exist in those economies. This result is reminiscent of the Modigliani-Miller theorem in corporate finance.

(2) *The structure of Optimal Capital and Labor Income Taxation.* There is an extensive literature in public finance on the structure of Ramsey taxation (see Diamond and Mirrles [15] and Atkinson and Stiglitz [2] and the references therein), and many have applied the theory of Ramsey taxation to the study of the structure of optimal capital and labor income taxation. Two questions that received most attentions are: (i) Should the tax on capital income be eliminated? (ii) Is uniform labor income taxation optimal? Among others, Boskin [8], Feldstein [16], and Atkinson and Sandmo [1] analyze the taxes on capital and labor income in life-cycle models with finite lives. They all find that the optimal capital income tax rates are, in general, *not zero.* As in static models, the structure of Ramsey taxation in these models depends on the structure of consumer's preference. However, Chamley [9] reexamines the problem in a general equilibrium model with infinite lives and finds that, for a wide class of utility functions (i.e. utility functions that are additively separable in time), the limiting optimal capital income tax rate *is zero* when the economy converges to a steady state. In this paper I study optimal taxation of capital and labor income in a model that is identical to Chamley's model except that the production technology and government expenditures are stochastic rather than deterministic. Two important properties about Ramsey taxation are found in this model. First, actual optimal capital income tax rates are indeterminate: there is an equivalent class of *ex post* actual capital income tax rates that lead to the same expected after tax return on capital. Therefore, it is important to distinguish between *ex post* actual capital income tax rates and *ex ante* expected capital income tax rates, and Ramsey's theory of optimal taxation only determines the *ex ante* expected capital income tax rate. Second, Chamley's result of zero limiting capital income tax rate

does not extend to the stochastic model. With uncertainty, the limiting behavior of optimal capital income tax rate again very much depends on the structure of consumer's preference. General conditions for the optimal ex ante expected capital income tax rates to be zero and for the optimal labor income tax rates to be constant are given as restrictions on the structure of consumer's preference. Examples of utility functions are also given to show that, in general, the optimal ex ante expected capital income tax rates may not be zero, and that uniform labor income taxation may not be optimal. Furthermore, for a class of utility functions that is commonly used in real business cycle models, it is shown that, in a stationary Ramsey equilibrium, the necessary and sufficient condition for the optimal ex ante expected capital income tax rates to be zero and for the optimal labor income tax rates to be constant across time and states is that employment rates are constant across time and states. Otherwise, the optimal ex ante expected capital income tax rates will be positive in some periods but negative in some other periods, and the optimal labor income tax rates will vary along with employment rates from period to period. In summary, the analyses in this paper indicate that there is no theoretical presumption that the optimal capital income tax rates should be zero and that uniform labor income taxation is optimal.

(3) *Dynamics of Ramsey Equilibrium.* After general analyses of the structure of Ramsey taxation, examples are given to describe the dynamics of Ramsey equilibrium. In these examples I find that the key variables determining the dynamics of a Ramsey equilibrium are the initial level of public debt outstanding and the time path of the ratios of government expenditures to output. By letting the government expenditures/output ratios follow different time paths, I analyze the corresponding time paths of the consumption/output ratios, the gross investment/output ratios, employment rates, and the optimal labor income tax rates. The main results that I find are that, first, both the consumption/output ratios and the gross investment/output ratios move inversely to the government expenditures/output ratios, whereas both employment rates and the optimal labor income tax rates move in the same direction as the government expenditures/output ratios. Second, the limiting level of the market value of optimal public issue is a function of both the initial level of public debt outstanding and the ratio of government expenditures to output in the initial period due to the fact that government spreads excess burden over time.

The paper is organized as follows. A model with endogenous capital accumulation and uncertainty is presented in Section 2. Competitive equilibria with labor and capital income taxes are then characterized in Section 3. This section also discusses the indeterminacy of capital income

tax rates and levels of public debt payments. The structure of optimal capital and labor income taxation is then analyzed in Section 4. Section 5 characterizes the dynamics of the Ramsey equilibrium in a simple economy, and Section 6 concludes the paper by discussing extensions of the results in this paper and directions for future researches.

2. The Model

The model that I use is a stochastic version of a one-sector neoclassical growth model augmented with a government sector. It is formulated in discrete time with an infinite horizon.

The economy is populated with many identical infinitely lived households. A typical household's preference over lifetime consumption and leisure is represented by

$$U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t), \qquad 0 < \beta < 1,$$

where c_t and l_t are the household's consumption and leisure time in period t, $u(\cdot, \cdot)$ is the period utility function, and β is the time discount rate. I assume that in each period the household is endowed with one unit of time, and that besides leisure time it devotes all of its remaining time to output production. Let n_t be the household's time spent in production, then, $l_t = 1 - n_t$.

In this economy there is only one final good which can either be consumed or invested. The production technology is represented by a constant return to scale production function

$$y_t = F(k_t, n_t; z_t),$$

where k_t and n_t are the capital and labor input in period t, and $\{z_t\}_{t \geq 0}$ is a given stochastic process representing shocks to the production technology.

There is a government in this economy. Government expenditure in units of consumption good in period t is denoted by g_t. Government expenditures are assumed to play no specific roles in the economy, and $\{g_t\}_{t \geq 0}$ is a given stochastic process. The only source of uncertainty in this economy is the fact that government expenditures and production technology are stochastic. So, in period t, the state of the economy is represented by a state variable $s^t = (s_0, s_2, ..., s_t)$, where $s_t = (g_t, r_t)$. I assume that all of the policy and price variables that will be specified later in this paper are functions of these state variables. For simplicity, I also assume that there are no shocks in the initial period or that households and government make all their decisions based on the realization of the shocks in the initial period.

Because the produced good can either be consumed or invested, the technology constraint is described by

$$g_t + c_t + k_{t+1} = F(k_t, n_t; z_t) + (1-\delta)\, k_t, \tag{2.1}$$

where δ is the rate of depreciation of capital.

The economy is decentralized with three perfectly competitive markets: the labor market, the capital market, and the government bond market. The capital market is a rental market where firms rent capital from consumers, and the government bond market is a contingent claim market where the government trades one-period-forward contingent claims with households. The timing of trading in these markets is important. I assume that in period t ($t>0$), both the capital market and the government bond market open before the shocks g_t and z_t are realized, whereas the labor market opens after the shocks are realized. All the results in this paper would carry through if one assumes that the capital market opens after the shocks are realized but the household's saving decisions have to be made before the shocks are realized.

It is well known that if the government uses a lump-sum tax to finance its expenditures, then the competitive equilibrium allocation is the first-best allocation. In this paper, however, I assume that government can only use flat-rate taxes on capital and labor income to finance its expenditures, and, following Ramsey's formulation, I assume that the government's objective is to find the optimal tax rates that induce the competitive allocation which maximizes the representative consumer's utility. I also assume that in each period the government can issue public debt or make public loans so that it can spread the excess burden across time. Thus a government's fiscal policy is a rule which specifies the state contingent tax rates and the amount of public debt payment in each period. I assume that the government announces its fiscal policy at the beginning of period zero and that it is able to precommit itself at time zero to the announced policy that will be carried out in later periods. In other words, I do not discuss time-inconsistency problems, which have been the focus of works by Fischer [17], Chari and Kehoe [11], Stockey [30], and others.

3. Competitive Equilibria with Flat-Rate Taxes

3.1. *Characterization of Competitive Equilibria*

In a perfectly competitive economy, the household takes as given the government's fiscal policy and all market prices when it makes its decisions. As a consumer, the household makes state contingent plans on consumption, labor supply, and savings in every period so that its expected lifetime utility is maximized. As the owner of a firm, the household hires the

optimal amount of capital and labor input so that its profit in each period is maximized. Since it is assumed that production technology is constant return to scale in capital and labor inputs, firms earn zero profits in equilibria.

The household has three ways of saving income: buying government bonds, investing in the capital market, or simply keeping its capital. As long as the present market value of after-tax return on capital is nonnegative, the household will never keep any amount of capital. However, if a capital income tax rate is so high that the present market value of the after-tax return on capital is negative, the household will keep the capital instead of investing it in the capital market. In this case, the government will not be able to collect any tax revenues from the household's capital income because there are none to collect. As a result, the government will always choose appropriate capital income tax rates such that the present market value of the after-tax return on capital is nonnegative in each period. More formally, in period t, the government will choose a capital income tax rate τ_t that satisfies the constraint

$$\int p_t(1-\tau_t)\, r_t\, ds_t \geqslant 0, \tag{3.1}$$

where r_t is the return on capital and p_t is the pricing density of one-period-forward contingent claims in period t.

Given the constraint (3.1), then, the household will always save its income by either buying government bonds or investing in capital market. In this case, the household's budget constraint is

$$c_t + k_{t+1} \leqslant (1-\theta_t)\, w_t n_t + (1-\delta+(1-\tau_t)\, r_t)\, k_t + b_t - \int p_{t+1} b_{t+1}\, dz_{t+1}, \tag{3.2}$$

where w_t is the wage rate, θ_t is the labor income tax rate, and b_t is the government's debt payment to the household, all in period t.

Thus, the *consumer's problem* is to maximize his/her expected utility under the budget constraint (3.2). The solution to this problem is characterized by the first order conditions

$$\frac{u_2(c_t, 1-n_t)}{u_1(c_t, 1-n_t)} = (1-\theta_t)\, w_t, \tag{3.3}$$

$$u_1(c_t, 1-n_t) = \beta E_t u_1(c_{t+1}, 1-n_{t+1})\, \bar{R}_{t+1}, \tag{3.4}$$

$$p_{t+1}(s_{t+1}, s^t) = \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \frac{u_1(c_{t+1}, 1-n_{t+1})}{u_1(c_t, 1-n_t)}, \quad \text{if} \quad \pi_t(s^t) > 0, \tag{3.5}$$

$$c_t + k_{t+1} = (1-\theta_t)\, w_t n_t + \bar{R}_t k_t + b_t - \int p_{t+1} b_{t+1}\, ds_{t+1}, \tag{3.6}$$

and the transversality conditions

$$\lim_{t \to \infty} \beta^t E_0 u_1(c_t, 1-n_t)\, b_t = 0, \tag{3.7}$$

$$\lim_{t \to \infty} \beta^t E_0 u_1(c_t, 1-n_t)\, \bar{R}_t k_t = 0, \tag{3.8}$$

where $\bar{R}_t = (1-\delta) + (1-\tau_t) r_t$ and $\pi_t(s^t)$ is the given probability density function of s^t.

Equations (3.3) and (3.4) are the marginal conditions for the household's consumption-leisure and consumption-investment choices; Eq. (3.5) equates the price of one-period-forward contingent claims in period t to the marginal rate of substitution between consumption in period t and consumption in period $t+1$, and Eq. (3.6) is the household's budget constraint which is binding for the optimal allocation. The transversality conditions (3.7) and (3.8) impose present value budget balance on the household.

The *firm's problem* is to maximize profits. Its solution is characterized by equating factor prices to the marginal products. That is,

$$r_t = F_1(k_t, n_t; z_t), \qquad w_t = F_2(k_t, n_t; z_t). \tag{3.9}$$

For any fiscal policy which satisfies constraint (3.1), then, the corresponding competitive equilibrium is characterized by conditions (3.1), (3.3) through (3.9) and the technology constraint (2.1). In this situation, the allocation $\{(c_t, n_t, k_{t+1})\}_{t \geq 0}$ is called a *competitive allocation* that can be *implemented* by the fiscal policy $\{\tau_t, \theta_t, b_{t+1}\}_{t \geq 0}$. It should be noted that, to prove that (2.1), (3.1), and (3.3) through (3.9) completely characterize competitive equilibria with flat-rate taxes, some Inada type conditions about the utility and production function are needed. A complete proof is available from the author upon request. In the rest of the paper, an allocation will be called a competitive allocation if it satisfies conditions (2.1), (3.1), and (3.3) through (3.9).

3.2. *Indeterminacy of Fiscal Policy*

In this stochastic economy with capital accumulation, an important feature of competitive equilibria with flat-rate taxes is that for any given competitive allocation there are an infinite number of different fiscal policies that can implement the allocation. In particular, capital income tax rates and levels of goverment debt payments that are consistent with certain competitive allocation are *indeterminate*. From the first-order condition (3.4) of the consumer's problem, one can see that capital income tax rates affect the consumer's intertemporal allocation by changing present market values of after-tax returns on capital. If different capital income tax rates induce the same present market value of after-tax return on capital,

then they will induce the same competitive allocation. Let the competitive allocation $\{c_t, n_t, k_{t+1}\}_{t\geqslant 0}$ be the one which can be implemented by certain fiscal policy $\{\tau_t, \theta_t, b_{t+1}\}_{t\geqslant 0}$. Let $\{\varepsilon_{t+1}\}_{t\geqslant 0}$ be a sequence of random variables such that

$$E_t u_1(c_{t+1}, 1-n_{t+1})\,\varepsilon_{t+1} r_{t+1} = 0, \qquad t \geqslant 0. \tag{3.10}$$

Define a new fiscal policy $\{\tau'_t, \theta'_t, b'_{t+1}\}_{t\geqslant 0}$ as follows:

$$\begin{aligned} \tau'_0 &= \tau_0, \\ \tau'_{t+1} &= \tau_{t+1} + \varepsilon_{t+1}, && t \geqslant 0, \\ \theta'_t &= \theta_t, && t \geqslant 0, \\ b'_{t+1} &= b_{t+1} + \varepsilon_{t+1} r_{t+1} k_{t+1}, && t \geqslant 0. \end{aligned}$$

Then the new fiscal policy can implement the same allocation as the original fiscal policy can. Since there are an infinite number of different sequences of random variables $\{\varepsilon_{t+1}\}_{t\geqslant 0}$ that satisfy (3.10), the allocation can be implemented by many different fiscal policies. In particular, one can choose an ε_{t+1} such that either the new capital income tax rate τ'_{t+1} is known in period t, or the new debt payment b'_{t+1} is known in period t. In other words, the government can use either state contingent capital income taxes and risk-free bonds or risk-free capital income taxes and state contingent bonds to implement the *same* competitive allocation.

Since every competitive allocation can be implemented by many different fiscal policies, the optimal allocation that maximizes the consumer's utility is also consistent with many different combinations of capital income tax rates and government debt payments. Thus, unlike the results in Barro [3], in this model the theory of optimal taxation theory does not determine the optimal levels of public debt payment in future periods. One of the key differences between this model and Barro's is that this model is a stochastic economy in which the government can issue contingent claim bonds.

Lucas and Stockey [26] analyze a stochastic economy without capital and find that the theory of optimal taxation does determine the optimal levels of public debt payment in future periods and that the use of state contingent bonds is important in achieving the Ramsey allocation. In their model, however, a capital market is missing and the tax on labor income is the only tax instrument available to the government. Unlike the tax on capital income which affects the household's behavior through some ex ante expected after-tax returns on capital, the tax on labor income affects the household's behavior directly through the actual realized tax rates. A critical difference between the tax on labor income and the tax on capital income is that the household can adjust the labor supply instantly when an

actual labor income tax rate is realized, but it cannot adjust the capital investment as quickly when an actual capital income tax rate is realized.

At this point, it may not be very clear what are the key features of the model that lead to the indeterminacy result and one may question the robustness of this result. In [33], however, I show that, as long as there exist complete capital markets (by which I mean there exist full contingent claim markets as well as rental markets), the indeterminacy result remains to hold even if there are heterogeneous consumers and production technologies. Furthermore, I show there that this indeterminacy result holds for any given transfer program that the government wants to implement. In other words, the indeterminacy result holds even if the economy consists of heterogeneous agents and different production technologies and the government is concerned about distributional effects of its fiscal policy. The most important property of the model that leads to the indeterminacy result is the requirement of the existence of complete capital markets. For that matter, this indeterminacy result can be viewed as an extension of the Modigliani–Miller theorem in corporate finance to public finance.

While levels of public debt payment in future periods and *ex post* capital income tax rates are indeterminate, the market values of public debt issue, the labor income tax rates, and the ex ante expected capital income tax rates that are consistent with certain competitive allocation are uniquely determined. Here the market value of public debt issue in period t is defined as

$$B_t = \int p_{t+1} b_{t+1}\, ds_{t+1} \tag{3.11}$$

and the ex ante expected capital income tax rate is defined as

$$\bar{\tau}_t = \frac{E_{t-1} u_1(c_t, 1-n_t)\, \tau_t r_t}{E_{t-1} u_1(c_t, 1-n_t)\, r_t}. \tag{3.12}$$

Using the first-order condition (3.5) to rewrite (3.12), we have

$$\bar{\tau}_t = \frac{\int p_t \tau_t r_t k_t\, ds_t}{\int p_t r_t k_t\, ds_t}$$

That is, the *ex ante* expected capital income tax rate is the ratio of present market value of tax revenue from capital income over the present market value of capital income. Thus, it can be interpreted as the *certainty equivalent* capital income tax rate.

In the remainder of this paper I study how labor income tax rates, ex ante expected capital income tax rates, and market values of public debt issue are optimally determined in a Ramsey equilibrium.

4. The Structure of Ramsey Taxation

4.1. *Ramsey Problem*

As been pointed out in Atkinson and Stiglitz [2], Lucas and Stockey [26], and Lucas [25], the Ramsey problem can be framed as to directly find the optimal *implementable* competitive allocation that maximizes the household's utility. So, the opportunity set of the government's optimization problem is the collection of all the implementable allocations—the allocations $\{c_t, n_t, k_{t+1}\}_{t\geqslant 0}$ that are consistent with some fiscal policy $\{\tau_t, \theta_t, b_{t+1}\}_{t\geqslant 0}$ in a competitive equilibrium for given k_0, b_0 and $\{g_t\}_{t\geqslant 0}$. The next proposition characterizes all such allocations.

PROPOSITION 1. *For given* k_0, b_0, *and* $\{g_t\}_{t\geqslant 0}$, *an allocation* $\{c_t, n_t, k_{t+1}\}_{t\geqslant 0}$ *is implementable if and only if there exists a* τ_0 *such that the following conditions are satisfied by the allocation*

$$\text{(i)}\quad \tau_0 \leqslant 1, \tag{4.1}$$

$$\text{(ii)}\quad \beta(1-\delta)\, E_t u_1(c_{t+1}, 1-n_{t+1}) \leqslant u_1(c_t, 1-n_t), \qquad \textit{for all}\quad t\geqslant 0, \tag{4.2}$$

$$\text{(iii)}\quad g_t + c_t + k_{t+1} \leqslant F(k_t, n_t; z_t) + (1-\delta)\, k_t, \qquad \textit{for all}\quad t\geqslant 0, \tag{4.3}$$

$$\text{(iv)}\quad u_1(c_0, 1-n_0)[b_0 + \bar{R}_0 k_0] = \sum_{t=0}^{\infty} E_0 \beta^t [u_1(c_t, 1-n_t)\, c_t - u_2(c_t, 1-n_t)\, n_t], \tag{4.4}$$

$$\text{(vi)}\quad \bar{R}_0 = 1 - \delta + (1-\tau_0)\, F_1(k_0, n_0; z_0), \tag{4.5}$$

$$\text{(viii)}\quad \lim_{t\to\infty} E_0 \beta^t u_1(c_{t-1}, 1-n_{t-1})\, k_t = 0. \tag{4.6}$$

Proof. See Appendix.

Constraints (4.1) and (4.2) are the constraints implied by the boundary condition (4) that is imposed on the capital income tax rates. Constraint (4.3) is the technology constraint. Constraint (4.4) is called the *implementability constraint*, which is the present value form of the government's budget constraint with the equilibrium prices substituted in Constraint (4.5) is just the definition of $\bar{R}_0$ and (4.6) is a transversality condition.

Thus, the Ramsey problem is to find the τ_0 and $\{c_t, n_t, k_{t+1}\}_{t\geqslant 0}$ that maximize the consumer's utility $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$ under the constraints (16) through (21) for given k_0, b_0, and $\{g_t\}_{t\geqslant 0}$.

PROPOSITION 2. *If* $\{c_t, n_t, k_{t+1}\}_{t\geqslant 0}$ *and* τ_0 *is a solution to the Ramsey problem, then either* (i) *it is the first best allocation, or* (ii) $\tau_0 = 1$ *and there exist a sequence* $\{\lambda_t, \mu_t\}_{t\geqslant 0}$ *and a positive constant* Φ *such that, for all* $t \geqslant 0$,

$$\lambda_t = \beta^t \pi_t(s^t)\, u_1(c_t, 1-n_t) \times [1 + \Phi \Lambda_t^c + \frac{u_{11}(c_t, 1-n_t)}{u_1(c_t, 1-n_t)} (\mu_t - (1-\delta)\,\mu_{t-1})], \tag{4.7}$$

$$\lambda_t = \frac{\beta^t \pi_t(s^t)\, u_2(c_t, 1-n_t)}{F_2(k_t, n_t; z_t)} \times [1 + \Phi \Lambda_t^n + \frac{u_{12}(c_t, 1-n_t)}{u_2(c_t, 1-n_t)} (\mu_t - (1-\delta)\,\mu_{t-1})], \tag{4.8}$$

$$\lambda_t = \int \lambda_{t+1} R_{t+1}\, ds_{t+1}, \tag{4.9}$$

$$\lambda_t \geqslant 0 \quad \text{and} \quad \mu_t \geqslant 0, \tag{4.10}$$

$$\lambda_t [F(k_t, n_t; z_t) + (1-\delta)\, k_t - g_t - c_t - k_{t+1}] = 0, \tag{4.11}$$

$$\mu_t [u_1(c_t, 1-n_t) - \beta(1-\delta)\, E_t u_1(c_{t+1}, 1-n_{t+1}))] = 0, \tag{4.12}$$

where μ_{-1} *is defined to be* 0,

$$\Lambda_t^c = \frac{u_{11}(c_t, 1-n_t)\, c_t - u_{12}(c_t, 1-n_t)\, n_t}{u_1(c_t, 1-n_t)} + 1, \quad \text{if} \quad t > 0,$$

$$\Lambda_0^c = \frac{u_{11}(c_0, 1-n_0)(c_0 - b_0 - (1-\delta)\, k_0) - u_{12}(c_0, 1-n_0)\, n_0}{u_1(c_0, 1-n_0)} + 1,$$

$$\Lambda_t^n = \frac{u_{12}(c_t, 1-n_t)\, c_t - u_{22}(c_t, 1-n_t)\, n_t}{u_2(c_t, 1-n_t)} + 1, \quad \text{if} \quad t > 0,$$

$$\Lambda_0^n = \frac{u_{12}(c_0, 1-n_0)(c_0 - b_0 - (1-\delta)\, k_0) - u_{22}(c_0, 1-n_0)\, n_0}{u_2(c_0, 1-n_0)} + 1,$$

and

$$R_{t+1} = 1 - \delta + F_1(k_{t+1}, n_{t+1}; z_{t+1}).$$

Proof. See Appendix.

There are two interesting aspects of the Ramsey solution in this economy, First, since the government can spread the excess burden over time by issuing public debt, the marginal excess burden in each period is

the same and is measured by the constant Φ, the Lagrangian multiplier associated with the implementability constraint. Second, since the supply of capital is inelastic in the initial period, the tax on capital income in that period is not distortionary. So the government wants to tax the capital income in the initial period as heavily as possible to minimize distortion caused by other distortionary taxes. If it happens that the revenue collected from this tax is big enough to finance all the current and future government expenditures, then there is no need to use distortionary taxes. Otherwise, the government will tax the capital income in the initial period at the highest possible rate, i.e., 100%, and finance the residual expenditures through distortionary taxes.

4.2. *Optimal Capital and Labor Income Taxation: General Results and Examples*

In this part of the section, I analyze the structure of optimal capital and labor income taxation in general. The conditions under which zero capital income tax rates and uniform labor income taxation are optimal will be discussed, and examples in which they are not optimal will also be given. The limiting structure of optimal capital and labor income taxation will be analyzed in the next part of the section.

To make analyses interesting, in the following I only consider the cases where the solution to the Ramsey problem is not a first-best solution. Therefore, given k_0, b_0, and $\{g_t\}_{t\geq 0}$, the Ramsey allocation is characterized by the first order conditions (4.7) through (4.12), constraints (4.2) through (4.6), and the boundary condition $\tau_0 = 1$. To simplify the analyses, I further assume, in this part of the section, that the boundary condition (4.2) in the Ramsey problem is never binding. This assumption will hold if either the depreciation rate of capital δ is very close to one or the amount of government expenditures in each period are small enough. Under this assumption, the Lagrangian multiplier associated with the boundary condition, μ_t, is zero in every period, and the first order conditions (4.7) through (4.12) are simplified to

$$\frac{u_2(c_t, 1-n_t)}{u_1(c_t, 1-n_t)} = \frac{1+\Phi\Lambda_t^c}{1+k\Lambda_t^n} F_2(k_t, n_t; z_t), \tag{4.13}$$

$$u_1(c_t, 1-n_t)(1+\Phi\Lambda_t^c) = \beta E_t u_1(c_{t+1}, 1-n_{t+1})(1+\Phi\Lambda_{t+1}^c) R_{t+1}, \tag{4.14}$$

$$R_{t+1} = 1-\delta+F_1(k_{t+1}, n_{t+1}, z_{t+1}). \tag{4.15}$$

Comparing these first-order conditions to the first-order conditions of the consumer's optimization problem, one can see that, in period t, the optimal labor income tax rate is

$$\theta_t = \frac{\Phi(\Lambda_t^n - \Lambda_t^c)}{1+\Phi\Lambda_t^n}, \tag{4.16}$$

and the optimal *ex ante* expected capital income tax rate is

$$\bar{\tau}_{t+1} = \frac{\Phi}{1+\Phi\Lambda_t^c} \frac{E_t u_1(c_{t+1}, 1-n_{t+1})\, R_{t+1}(\Lambda_t^c - \Lambda_{t+1}^c)}{E_t u_1(c_{t+1}, 1-n_{t+1})\, F_1(k_{t+1}, n_{t+1}; z_{t+1})}. \quad (4.17)$$

Before going into specific analyses of the structure of optimal capital and labor income taxation, it is useful to compare the tax structure in this dynamic model to those in static models of commodity taxation. In this model, taxes are not imposed on consumption and leisure—the commodities in this economy. But these two commodities are indirectly taxed by the capital and labor income tax. In fact, it can easily be shown that the income tax system in this model is equivalent to a tax system which has no capital income tax but imposes a consumption tax in every period except the initial period, and that zero capital income tax rate in period t is equivalent to having the same consumption tax rate in period t and period $t+1$. Using this analogy between capital income taxation and consumption taxation, then, one can infer that the determinants of the structure of optimal capital income taxation should be similar to the determinants of the structure of optimal commodity taxation in static models. In those models, the determinants of the structure of optimal taxation are the elasticities of the so called *compensated inverse demand* functions of taxed goods with respect to an untaxed good (usually leisure). (For general discussions about compensated inverse demand functions and their applications in the theory of optimal taxation, see Deaton [12, 13] and Deaton and Muellbauer [14]). In this model, the untaxed good is consumption in period zero. Therefore, one would expect that the structure of optimal capital income taxation depends on the elasticities of compensated inverse demand functions of consumption in future periods with respect to consumption in period zero. The next proposition shows that this is indeed the case. Furthermore, the proposition also shows that the structure of optimal labor income taxation depends on the elasticities of compensated inverse *supply* functions of labor with respect to consumption in period zero.

For any constant $\bar{u}$ and a sequence of consumption and labor supply $\{c_t, n_t\}_{t\geq 0}$, define the value of the *distance* function $d(\bar{u}, c_t, n_t, t \geq 0)$ as

$$\inf\left\{\lambda \geq 0 : E_0 \sum_{t=0}^{\infty} \beta^t u(c_t/\lambda, 1-n_t/\lambda) \geq \bar{u}\right\},$$

and define the compensated inverse demand function of consumption and the compensated inverse supply function of labor as follows:

$$a_t^c = \frac{\partial d}{\partial c_t}, \quad a_t^n = -\frac{\partial d}{\partial n_t}, \qquad t \geq 0.$$

We have

PROPOSITION 3. *Let* $\{c_t, n_t\}_{t\geqslant 0}$ *be a sequence of consumption and labor supply that satisfies the implementability constraint* (4.4). *Let* $\bar{u}$ *be the expected utility level that is achieved by the sequence. Then, at* $\bar{u}$,

$$\Lambda_t^c - \Lambda_{t+1}^c = (b_0 + \bar{R}_0 k_0) \frac{\partial \log(a_{t+1}^c/a_t^c)}{\partial c_0}, \tag{4.18}$$

$$\Lambda_t^n - \Lambda_t^c = (b_0 + \bar{R}_0 k_0) \frac{\partial \log(a_t^c/a_t^n)}{\partial c_0} \tag{4.19}$$

for all $t \geqslant 0$.

Proof. See Appendix.

Substituting (4.18) into (4.17), then, we have that, in a Ramsey equilibrium where $b_0 + (1-\delta) k_0 \neq 0$, $\bar{\tau}_{t+1} = 0$ if and only if

$$\frac{\partial \log(a_t^c)}{\partial c_0} = \frac{E_t u_1(c_{t+1}, 1-n_{t+1}) R_{t+1} (\partial \log(a_{t+1}^c)/\partial c_0)}{E_t u_1(c_{t+1}, 1-n_{t+1}) R_{t+1}}.$$

That is, the condition for zero capital income tax rate in period t (or constant consumption rates in period t and $t+1$) is that the compensated inverse demand elasticity of consumption in period t equals the average compensated inverse demand elasticity of consumption in period $t+1$. This result is reminiscent of Deaton's [13] result in commodity taxation literature that two goods bear the same tax rate if and only if their elasticities of compensated inverse demand with respect to the untaxed good are the same. Similarly, the results in commodity taxation literature about uniform taxation can also be applied to the analysis of optimal labor income taxation in this model. One of the commonly used conditions for uniform commodity taxation is that utility is homogeneously separable. In this model, we have

PROPOSITION 4. *Assume that the boundary condition* (4.2) *is never binding. Then*

(i) $\bar{\tau}_{t+1} = 0$ *for all* $t > 0$ *if* $u(c, 1-n) = v(c, n) + \varphi(1-n)$ *and* v *is homogeneous in* c *and* n;

(ii) θ_t *is constant for all* $t > 0$ *if* $u(c, 1-n)$ *is homogeneous in* c *and* n *or* $u(c, 1-n) = 1/(1-\sigma)\, c^{1-\sigma} - 1/(1-\gamma)\, n^{1-\gamma}$ *for some* $\sigma > 0$ *and* $\gamma > 0$.

Proof. See Appendix.

Chamley [9] shows that if there are no uncertainties and if the period utility function is such that $u(c, 1-n) = 1/(1-\sigma)\, c^{1-\sigma} + \varphi(1-n)$, then the optimal capital income tax rates are zero. Proposition 4 extends his result

to the stochastic case and to a more general class of utility functions. It also gives conditions for uniform labor income taxation to be optimal. Next, through examples I show that for some utility functions that do not satisfy conditions in (ii) and (iii), the optimal capital income tax rate may not be zero and uniform labor income taxation may not be optimal.

A CLASS OF UTILITY FUNCTIONS. Consider a class of utility functions that is often used in real business cycle models:

$$u(c, l)=\frac{1}{1-\sigma}(c\varphi(l))^{1-\sigma}, \qquad \textit{for} \quad 0<\sigma<1, \quad \text{or} \quad \sigma>1. \tag{4.20}$$

For these utility functions, we have

$$\Lambda_t^c=(1-\sigma)\left[1-\frac{\varphi'(1-n_t)}{\varphi(1-n_t)}\right], \tag{4.21}$$

$$\Lambda_t^n=2-\sigma-\left[\frac{\varphi''(1-n_t)}{\varphi'(1-n_t)}-\sigma\frac{\varphi'(1-n_t)}{\varphi(1-n_t)}\right]n_t \tag{4.22}$$

for every $t>0$. From concavity of the period utility function $u(\cdot,\cdot)$, we have $u_{22}(c, l)<0$ and $u_{11}(c, l)\,u_{22}(c, l)-u_{12}^2(c, l)>0$. Thus, for the utility functions defined in (4.20), we have

$$\sigma\varphi(l)\,\varphi''(l)+(1-2\sigma)\,\varphi'^2(l)>0,$$

$$\varphi(l)\,\varphi''(l)-\sigma\varphi'^2(l)<0.$$

Combining these two inequalities, we have

$$(1-\sigma)(\varphi(l)\,\varphi''(l)-\varphi'^2(l))<0.$$

But from (4.21),

$$d\Lambda_t^c/dl_t=-(1-\sigma)\frac{\varphi(l)\,\varphi''(l)-\varphi'^2(l)}{\varphi'^2(l)}.$$

Thus, $d\Lambda_t^c/dl_t>0$ or $d\Lambda_t^c/dn_t<0$, i.e., Λ_t^c is a strictly monotonic function of n_t.

EXAMPLE 4.1. Assume that there are no uncertainties. From (4.17), the optimal capital income tax rate in period $t+1$ is

$$\tau_{t+1}=\frac{\Phi R_{t+1}}{(1+\Phi\Lambda_t^c)\,F_1(k_{t+1}, n_{t+1}; z_{t+1})}(\Lambda_t^c-\Lambda_{t+1}^c).$$

So $\tau_{t+1}=0$ if and only if $\Lambda_t^c=\Lambda_{t+1}^c$. Since Λ_t^c is a strictly monotonic function of n_t, in this example, the optimal capital income tax rates are zero if and only if employment rates stay constant over time.

EXAMPLE 4.2. Assume that $\varphi(x)=x^\alpha$. From (4.16), the optimal labor income tax rate period t is simply

$$\theta_t=\frac{\Phi}{1+\Phi(2-\sigma)-[1+\Phi(1+\alpha)(1-\sigma)]\,n_t} \tag{4.23}$$

and $d\theta_t/dn_t>0$ if $\sigma\leqslant 1$. That is, for $\sigma\leqslant 1$, the optimal labor income tax rates and employment rates move in the same direction, and uniform labor income taxation is not optimal unless employment rates stay constant across time and states.

Example 4.1 and 4.2 indicate that, for some commonly used utility functions, the condition for the optimal *ex ante* expected capital income tax rates to be zero and for uniform labor income taxation to be optimal is that employment rates stay constant across time and states, a very restrictive condition. (In fact, with utility functions defined in (4.20), an example can easily be constructed in which employment rates are not constant in the Ramsey equilibrium, see Zhu [33, Appendix B]). In the literature of dynamic Ramsey taxation some have argued that uniform labor income taxation is optimal (see Barro [3] and Kydland and Prescott [24]). The analyses in this section show that there is no theoretical presumption that uniform labor income taxation is an optimal policy. The optimality of the policy depends on the structure of consumer's preference.

Optimal capital income tax rates may never be zero. But Chamley [9] shows that, when there are no uncertainties and when the Ramsey equilibrium converges to a deterministic steady state, the limiting optimal capital income tax rate is zero *independent* of specifications of consumer's utility function (except that it is required that utility function is additively separable in time). This is so because the elasticities of compensated inverse demand functions of consumption converge to a constant when the economy converges to a deterministic steady state. In other words, as the economy converges to a deterministic steady state, the consumption demand becomes symmetric over time, therefore, it is not optimal to have a discriminatory consumption tax. Since taxes on capital income tax future consumption more heavily than current consumption, it is optimal to set limiting capital income tax rates to zero. A natural question to ask is, then, does Chamley's result of zero limiting capital income tax rate extends to stochastic models as well? It is this question that the rest of this section turns to.

4.3. *Optimal Capital and Labor Income Taxation: Stationary Equilibria*

In this stochastic economy, a Ramsey equilibrium is defined as a stationary equilibrium if it satisfies the following conditions:

(i) The shocks to the economy form a first-order Markov process with a stationary transition density function $\pi(\cdot,\cdot)\colon \mathbf{S}^2 \to [0,1]$, where $\mathbf{S}$ is a compact set.

(ii) The consumer's aggregate consumption, working time, and investment are all time-invariant functions of the state variables k and s, where k is the current capital stock and s is the current shock to the economy. That is,

$$c_t = c(k_t, s_t), \qquad n_t = n(k_t, s_t), \qquad \text{and} \qquad k_{t+1} = \phi(k_t, s_t).$$

(iii) The stochastic process $\{k_t, s_t\}_{t\geqslant 0}$ is a stationary, ergodic, first-order Markov process. That is, there exists a probability measure $P^{\infty}\{\cdot\}$ on $\mathscr{P} = B([0,\bar{k}]) \times B(\mathbf{S})$ such that, for any $A \in \mathscr{P}$,

$$\text{Prob}\{(k_t, s_t) \in A\} = P^{\infty}\{A\}, \qquad \text{for all } t;$$

$$\lim_{j \to +\infty} \text{Prob}\{(k_{t+j}, s_{t+j}) \in A \mid k_t, s_t\} = P^{\infty}\{A\}.$$

Here $\bar{k}$ is the maximal sustainable capital stock.

To avoid some pathological cases, I make two regularity assumptions:

Assumption 1. For every closed set A in $[0,\bar{k}] \times \mathbf{S}$, $t>0$ and $j>0$, $\text{Prob}\{(k_{t+j}, s_{t+j}) \in A \mid k_t, s_t\}$ is a continuous function of k_t and s_t.

Assumption 2. The policy functions $c(\cdot,\cdot)$, $n(\cdot,\cdot)$, and $\phi(\cdot,\cdot)$ are all continuous.

Under these two assumptions, we have:

PROPOSITION 5. *If a Ramsey equilibrium is stationary, then,*

(i) *either* $P^{\infty}\{\bar{\tau}_t = 0\} = 1$, *or* $P^{\infty}\{\bar{\tau}_t > 0\} > 0$ *and* $P^{\infty}\{\bar{\tau}_t < 0\} > 0$;

(ii) $P^{\infty}\{\bar{\tau}_t = 0\} = 1$ *if and only if* $P^{\infty}\{A_t^c = \bar{A}\} = 1$ *for some constant* $\bar{A}$;

(iii) *when* $b_0 + (1-\delta)k_0 \neq 0$, $P^{\infty}\{\bar{\tau}_t = 0\} = 1$ *if and only if* $P^{\infty}\{\partial \log(a_t^c)/\partial c_0 = \bar{a}\} = 1$ *for some constant* $\bar{a}$.

Proof. See Appendix.

Result (i) of Proposition 5 says that, in a stationary Ramsey equilibrium, the optimal ex ante expected capital income tax rates cannot be always positive nor always negative. Either they are zero with probability one or the probabilities of them being positive and negative are both greater than

zero. Results (ii) and (iii) of Proposition 5 give necessary and sufficient conditions under which the optimal ex ante expected capital income tax rates are zero with probability one. Evidently, if the period utility function $u(\cdot, \cdot)$ satisfies conditions in Proposition 4 or the Ramsey equilibrium is in a determinstic steady state, then the conditions in (ii) and (iii) are satisfied and the optimal ex ante expected capital income tax rates are zero with probability one. However, for the utility functions defined in (4.20), it is shown that Λ_t^c is a strictly monotonic function of n_t. Therefore, for this class of utility functions, the optimal ex ante expected capital income tax rates are zero with probability one if and only if employment rates are constant with probability one. Thus, Chamley's result of zero limiting capital income tax rate does not extend to this stochastic model. In stationary Ramsey equilibria, the zeroness of optimal ex ante capital income tax rates depends on the specification of consumer's preference, as indicated by conditions (ii) and (iii) of Proposition 5.

5. Dynamic Characterization of the Ramsey Equilibrium in a Simple Economy

This section presents a simple economy in which one can characterize the dynamic structure of the Ramsey equilibrium. To make the analysis simple and tractable, the following assumptions are made:

(i) logarithmic utility function: $u(c_t, 1-n_t) = \log(c_t) + \alpha \log(1-n_t)$;

(ii) Cobb–Douglas production function: $y_t = F(k_t, n_t; z_t) = k_t^{\varepsilon} n_t^{1-\varepsilon} m(z_t)$;

(iii) full depreciation: $\delta = 1$.

Under these assumptions, the Ramsey allocation is characterized by the following conditions:

first order conditions:

$$\frac{\alpha c_t}{1-n_t} = \frac{1-n_t}{\Phi + 1 - n_t}(1-\varepsilon)\, k_t^{\varepsilon} n_t^{-\varepsilon} m(z_t), \qquad t > 0, \quad (5.1)$$

$$\frac{\alpha c_0}{1-n_0}\left(1 + \frac{\Phi b_0}{c_0}\right)^{-1} = \frac{1-n_0}{\Phi + 1 - n_0}(1-\varepsilon)\, k_0^{\varepsilon} n_0^{-\varepsilon} m(z_0), \quad (5.2)$$

$$c_t^{-1} = \varepsilon\beta E_t c_{t+1}^{-1} k_{t+1}^{\varepsilon-1} n_{t+1}^{1-\varepsilon} m(z_{t+1}), \qquad t > 0, \quad (5.3)$$

$$c_0^{-1}\left(1 + \frac{\Phi b_0}{c_0}\right) = \varepsilon\beta E_0 c_1^{-1} k_1^{\varepsilon-1} n_1^{1-\varepsilon} m(z_1); \quad (5.4)$$

technology constraint:

$$g_t + c_t + k_{t+1} = k_t^{\varepsilon} n_t^{1-\varepsilon} m(z_t), \qquad t \geqslant 0; \tag{5.5}$$

implementability constraint:

$$c_0^{-1} b_0 = \sum_{t=0}^{\infty} E_0 \beta^t \left(1 - \frac{\alpha n_t}{1 - n_t}\right); \tag{5.6}$$

transversality condition:

$$\lim_{t \to \infty} E_0 \beta^t \frac{k_{t+1}}{c_t} = 0. \tag{5.7}$$

Let ω_t denote the ratio of government expenditures to output g_t/y_t and a_t the ratio of consumption to output c_t/y_t. From (5.1) through (5.7), for every $t > 0$,

$$c_t/y_t = a_t = \left[E_t \sum_{j=0}^{\infty} \frac{(\varepsilon\beta)^j}{(1-\omega_t)\cdots(1-\omega_{t+j})}\right]^{-1}, \tag{5.8}$$

$$n_t = n(\Phi, a_t) = 1 - \frac{(1-\Phi)\,\alpha a_t + [4(1-\varepsilon)\,\alpha a_t \Phi + a^2 a_t^2 (1+\Phi)^2]^{1/2}}{2(1-\varepsilon+\alpha a_t)}, \tag{5.9}$$

$$k_{t+1}/y_t = 1 - \omega_t - a_t, \tag{5.10}$$

and the values of a_0, n_0 and Φ are determined by the following equations:

$$\frac{\alpha a_0}{1-n_0} = \frac{1-n_0}{\Phi + 1 - n_0}\,(1-\varepsilon)\,n_0^{-1}\left(1 + \frac{\Phi b_0}{a_0 k_0^{\varepsilon} n_0^{1-\varepsilon}}\right), \tag{5.11}$$

$$a_0^{-1}\left(1 + \frac{\Phi b_0}{a_0 k_0^{\varepsilon} n_0^{1-\varepsilon}}\right)(1 - \omega_0 - a_0) = E_0 \sum_{j=0}^{\infty} \frac{(\varepsilon\beta)^{j+1}}{(1-\omega_1)\cdots(1-\omega_{j+1})}, \tag{5.12}$$

$$\frac{b_0}{a_0 k_0^{\varepsilon} n_0^{1-\varepsilon}} = 1 - \alpha \frac{n_0}{1-n_0} + E_0 \sum_{t=1}^{\infty} \beta^t \left[1 - \frac{\alpha n(\Phi, a_t)}{1 - n(\Phi, a_t)}\right] = 0. \tag{5.13}$$

Correspondingly, the optimal *ex ante* expected capital income tax rates, the optimal labor income tax rates, and the market values of public debt issue can be written as follows:

the optimal labor income tax rates:

$$\theta_t = \frac{\Phi}{\Phi + 1 - n(\Phi, a_t)}, \qquad t > 0; \tag{5.14}$$

the optimal ex ante expected capital income tax rates:

$$\bar{\tau}_0 = 1, \tag{5.15}$$

$$\bar{\tau}_t = 0, \qquad t > 1; \tag{5.16}$$

the market values of the optimal public debt issue:

$$B_t = \left[-(1-\omega_t) + a_t + a_t E_t \sum_{j=1}^{\infty} \beta^j \left(1 - \frac{\alpha n_{t+j}}{1-n_{t+j}} \right) \right] y_t, \qquad t > 0. \tag{5.17}$$

From Eqs. (5.8) through (5.17), one can see that in the Ramsey equilibrium the consumption/output ratios, the gross investment/output ratios, the employment rates, the optimal labor income tax rates, and the ratios of market values of public debt issue to output are all functions of the ratios of government expenditures to output and the constant Φ. But from (5.11) through (5.13), the constant Φ itself is a function of initial values b_0, ω_0 and the initial distribution of ω_t's. Therefore, the limiting behavior of the Ramsey equilibrium depends on these initial values as well as the initial distribution of ω_t's. In particular, the limiting ratio of the market value of public debt issue to output is a function of the initial level of public debt outstanding b_0. This dependence comes from the fact that the optimizing government is trying to spread excess burden over time and states. When the government's obligation in the initial period is high it will increase the level of tax rates in *every* period and state, not just in the initial period. As a result, the limiting level of public debt issue is a function of the initial level of the public debt outstanding as well as the initial level of government expenditures to output ratio.

For simplicity, in the rest of this section I assume that there is no public debt outstanding in the initial period, i.e., $b_0 = 0$. In this case, we have

$$a_0 = \left[E_0 \sum_{j=0}^{\infty} \frac{(\varepsilon\beta)^j}{(1-\omega_0)\cdots(1-\omega_j)} \right]^{-1}, \tag{5.18}$$

$$n_0 = n(\Phi, a_0) = 1 - \frac{(1-\Phi)\,\alpha a_0 + [4(1-\varepsilon)\,\alpha a_0 \Phi + \alpha^2 a_0^2 (1+\Phi)^2]^{1/2}}{2(1-\varepsilon+\alpha a_0)}, \tag{5.19}$$

$$k_1/y_0 = 1 - \omega_0 - a_0, \tag{5.20}$$

and the value of Φ is determined by the condition

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[1 - \frac{\alpha a(\Phi, a_t)}{1 - n(\Phi, a_t)} \right] = 0. \tag{5.21}$$

Also, we have $\bar{\tau}_1 = 0$ and

$$\theta_0 = \frac{\Phi}{\Phi + 1 - n(\Phi, a_0)}. \tag{5.22}$$

Thus, the Ramsey equilibrium is completely determined by the sequence of government expenditures to output ratios. In other words, if one can determine the equilibrium time path of the government/ouptput ratio, the time paths of all other economic variables can be determined through Eq. (5.8) through (5.17). In the following examples I examine how various economic variables vary over time when the government expenditures/output ratios follow certain time paths.

Since it has been assumed that government expenditures are exogenously given, the government expenditures/output ratios are, then, endogenously determined in the equilibrium. In other words, one cannot simply *pick* arbitrary time path for the government expenditures/output ratios. However, there are two justifications for the experiments that I am going to conduct here. First, for any given stochastic process $\{\omega_t\}_{t\geqslant 0}$, if it satisfies the conditions

$$\omega_t > 0, \qquad t \geqslant 0,$$

$$1-\omega_t-\left[E_t\sum_{j=0}^{\infty}\frac{(\varepsilon\beta)^j}{(1-\omega_t)\cdots(1-\omega_{t+j})}\right]^{-1}>0, \qquad t\geqslant 0$$

then, by defining government expenditures $\{g_t\}_{t\geqslant 0}$ through the equations

$$\log(g_t)=\varepsilon^{t+1}\log(k_0)+V(a_t,\omega_t,z_t)+\cdots+\varepsilon^t V(a_0,\omega_0,z_0)$$
$$+\log(\omega_t)-\log(1-\omega_t-a_t),$$

$$V(a,\omega,z)=(1-\varepsilon)\log(n(\Phi,a))+\log(m(z))+\log(1-\omega-a),$$

$$a_t=\left[E_t\sum_{j=0}^{\infty}\frac{(\varepsilon\beta)^j}{(1-\omega_t)\cdots(1-\omega_{t+j})}\right]^{-1},$$

one can easily verify that, in the Ramsey equilibrium, the government expenditures/output ratio in period t is simply ω_t. Thus, for each of the experiments that I conduct below (in which the condition on ω_t's I specified above is always satisfied), there always exists a corresponding sequence of government expenditures that would give rise the stipulated government expenditures/output ratios in a Ramsey equilibrium. These experiments are similar to those done by Grossman [18] in a model of monetary equilibrium. He lets interest rate, which is an endogenous variable, follow certain time paths, and then determines corresponding money supply policies that are consistent with the chosen interest rate in equilibrium.

The second justification for the experiments is that, in [33], an example is given which shows that when government expenditures are public goods

that enter into private production as an input the optimal levels of government expenditures will be chosen such that the ratios of them to output equals the share of public goods in private production. Therefore, the following examples can also be interpreted as characterization of dynamics of the competitive equilibrium when both tax rates and the levels of government expenditures are determined endogenously.

The following properties of the employment rate and the optimal labor income tax rate are used in all of the examples below: For given Φ, (i) the employment rate n_t is a decreasing function of a_t, and (ii) the optimal labor income tax rate θ_t is an increasing function of n_t.

EXAMPLE 5.1. Let $\omega_t = \bar{\omega} > 0$, $\bar{\omega} < 1 - \varepsilon\beta$, for all $t \geqslant 0$. From (5.8) and (5.10), $c_t/y_t = 1 - \bar{\omega} - \varepsilon\beta$, and $k_{t+1}/y_t = \varepsilon\beta$. Since $a_t = c_t/y_t$ is constant over time, from (5.9) n_t is also constant over time. From (5.21), then, $n_t = 1/(1+\alpha)$. Finally, from (5.14) and (5.17) $\theta_t = [\bar{\omega} + \varepsilon(\beta - 1)]/(1-\varepsilon)$ and $B_t/y_t = -\varepsilon\beta$.

When government expenditures/output ratios are constant, the optimal labor income tax rates are constant over time. In this case, the optimal fiscal policy is as follows: (i) in the initial period, the government taxes both labor and capital income, runs a surplus, and uses the surplus to buy bonds from the public; (ii) starting from the second period, the government stops taxing capital income and finances its expenditures by the tax on labor income and the interest income from the bonds it bought from the public in the initial period.

Note that, in this example, market values of public debt issue are not necessarily constant over time. They vary with the level of output. As a result, the deficit is not zero in every period even if the ratios of government expenditures to output are constant. In fact, the deficit–output ratios depends on the change in the output:

$$\frac{B_t - B_{t-1}}{y_t} = \varepsilon\beta\left(\frac{y_{t-1}}{y_t} - 1\right).$$

EXAMPLE 5.2. Let $\{\omega_t\}_{t \geqslant 0}$ be a sequence of independent and identically distributed random variables, $E(1-\omega_t)^{-1} < \varepsilon\beta$. From (5.8),

$$a_t = (1-\omega_t)\left(1 - \varepsilon\beta E\frac{1}{1-\omega_t}\right), \qquad \text{for} \quad t \geqslant 0. \tag{5.23}$$

Substituting a_t into (5.10) and (5.17), one has that

$$k_{t+1}/y_t = (1-\omega_t)\,\varepsilon\beta E\frac{1}{1-\omega_t}, \qquad \text{for} \quad t \geqslant 0, \tag{5.24}$$

and that

$$B_t/y_t = (1-\omega_t)\left[-1+\left(1-\varepsilon\beta E\frac{1}{1-\omega_t}\right)\left(1+\frac{\beta}{1-\beta}E\left(1-\alpha\frac{n_t}{1-n_t}\right)\right)\right],$$

$$\text{for} \quad t \geqslant 0. \qquad (5.25)$$

Since $E(1/(1-\omega_t))$ and $E(n_t/(1-n_t))$ are both constants, Eq. (5.23) through (5.25) imply that $\{a_t\}_{t\geqslant 0}$, $\{k_{t+1}/y_t\}_{t\geqslant 0}$ and $\{B_t/y_t\}_{t\geqslant 0}$ are sequences of independent and identically distributed random variables, and consequently, from (5.9) and (5.14), $\{n_t\}_{t\geqslant 0}$ and $\{\theta_t\}_{t\geqslant 0}$ are also sequences of independent and identically distributed random variables. Furthermore, from (5.23) and (5.24), both the ratio of consumption to output and the ratio of gross investment to output are decreasing functions of the government expenditures/output ratio ω_t. From (5.9) and (5.14), then, both employment rates and the optimal labor income tax rates are increasing functions of the government expenditures/output ratio ω_t.

EXAMPLE 5.3. For some $T>0$, let $\omega_t = \bar{\omega} < 1-\varepsilon\beta$ when $t \neq T$, and $\omega_T = \bar{\bar{\omega}} > \bar{\omega}$ (i.e., there is a temporary increase in the ratio of the government expenditures to output). Let $\gamma = \varepsilon\beta/(1-\bar{\omega})$, and assume that $\gamma < 1$. Then, from (5.8),

$$a_t = \frac{(1-\bar{\bar{\omega}})(1-\bar{\omega}-\varepsilon\beta)}{1-\bar{\bar{\omega}}+(\bar{\bar{\omega}}-\bar{\omega})\gamma^{T-t}}, \qquad 0 \leqslant t \leqslant T \qquad (5.26)$$

and

$$a_t = (1-\bar{\omega}-\varepsilon\beta), \qquad t > T. \qquad (5.27)$$

Substituting (5.26) and (5.27) into (5.10), one has that

$$k_{t+1}/y_t = \frac{\varepsilon\beta[1-\bar{\bar{\omega}}+(\bar{\bar{\omega}}-\bar{\omega})\gamma^{T-t-1}]}{1-\bar{\bar{\omega}}+(\bar{\bar{\omega}}-\bar{\omega})\gamma^{T-t}}, \qquad 0 \leqslant t < T, \qquad (5.28)$$

$$k_{T+1}/y_T = \frac{1-\bar{\bar{\omega}}}{1-\bar{\omega}}\varepsilon\beta, \qquad (5.29)$$

and

$$k_{t+1}/y_t = \varepsilon\beta, \qquad t > T. \qquad (5.30)$$

From (5.26) through (5.30), the movements of $\{a_t\}_{t\geqslant 0}$ and $\{k_{t+1}/y_t\}_{t\geqslant 0}$ can be described as follows. Starting from the initial period, a_t decreases till period T, jumps up to $1-\bar{\omega}-\varepsilon\beta$ in period $T+1$, and stays constant thereafter. k_{t+1}/y_t increases till period $T-1$, drops in period T, increases again in period $T+1$, and stays constant thereafter. From (5.9) and (5.14), then, both the employment rate and the optimal labor income tax rate increase till period T, jump down in period $T+1$, and stay constant thereafter. The movements of all these variables are plotted in Fig. 1 with $\bar{\omega}=0.2$ and $\bar{\bar{\omega}}=0.25$. The parameters ε, α, and β are set to 0.25, 0.5, and 0.99 in the plots. Since the annualized value of β is estimated to be around 0.97, one time period here can be interpreted as one quarter of a year. In the plots T is set to be 20. The value of Φ is determined by solving Eq. (5.21) using the Newton–Raphson method.

EXAMPLE 5.4. For some $T>0$, let $\omega_t=\bar{\omega}$ when $t<T$, and $\omega_t=\bar{\bar{\omega}}>\bar{\omega}$ when $t\geqslant T$ (i.e., there is a permanent increase in the ratio of the government expenditures to output). Assume that $\bar{\bar{\omega}}<1-\varepsilon\beta$. Then, from (5.8),

$$a_t=\frac{(1-\bar{\omega}-\varepsilon\beta)(1-\bar{\bar{\omega}}-\varepsilon\beta)}{1-\bar{\bar{\omega}}-\varepsilon\beta+(\bar{\bar{\omega}}-\bar{\omega})\gamma^{T-t}}, \qquad 0\leqslant t<T, \tag{5.31}$$

and

$$a_t=1-\bar{\bar{\omega}}-\varepsilon\beta, \qquad t\geqslant T. \tag{5.32}$$

Substituting (5.31) and (5.32) into (5.10), one has that

$$k_{t+1}/y_t=\frac{\varepsilon\beta[1-\bar{\bar{\omega}}-\varepsilon\beta+(\bar{\bar{\omega}}-\bar{\omega})\gamma^{T-t-1}}{1-\bar{\bar{\omega}}-\varepsilon\beta+(\bar{\bar{\omega}}-\bar{\omega})\gamma^{T-t}}, \qquad 0\leqslant t<T, \tag{5.33}$$

and

$$k_{t+1}/y_t=\varepsilon\beta, \qquad t\geqslant T. \tag{5.34}$$

In this example, a_t decreases till period T and stays constant thereafter. Consequently, employment rates and the optimal labor income tax rates increase till period T and stay constant thereafter. k_{t+1}/y_t increases till period $T-1$ and then drops in period T and stays constant thereafter. The movements of these variables are plotted in Fig. 2 with $\bar{\omega}=0.15$ and $\bar{\bar{\omega}}=0.20$. The values of ε, α, β, and T are the same as in Example 5.3, and the value of Φ are determined by solving Eq. (5.21) using the Newton–Raphson method.

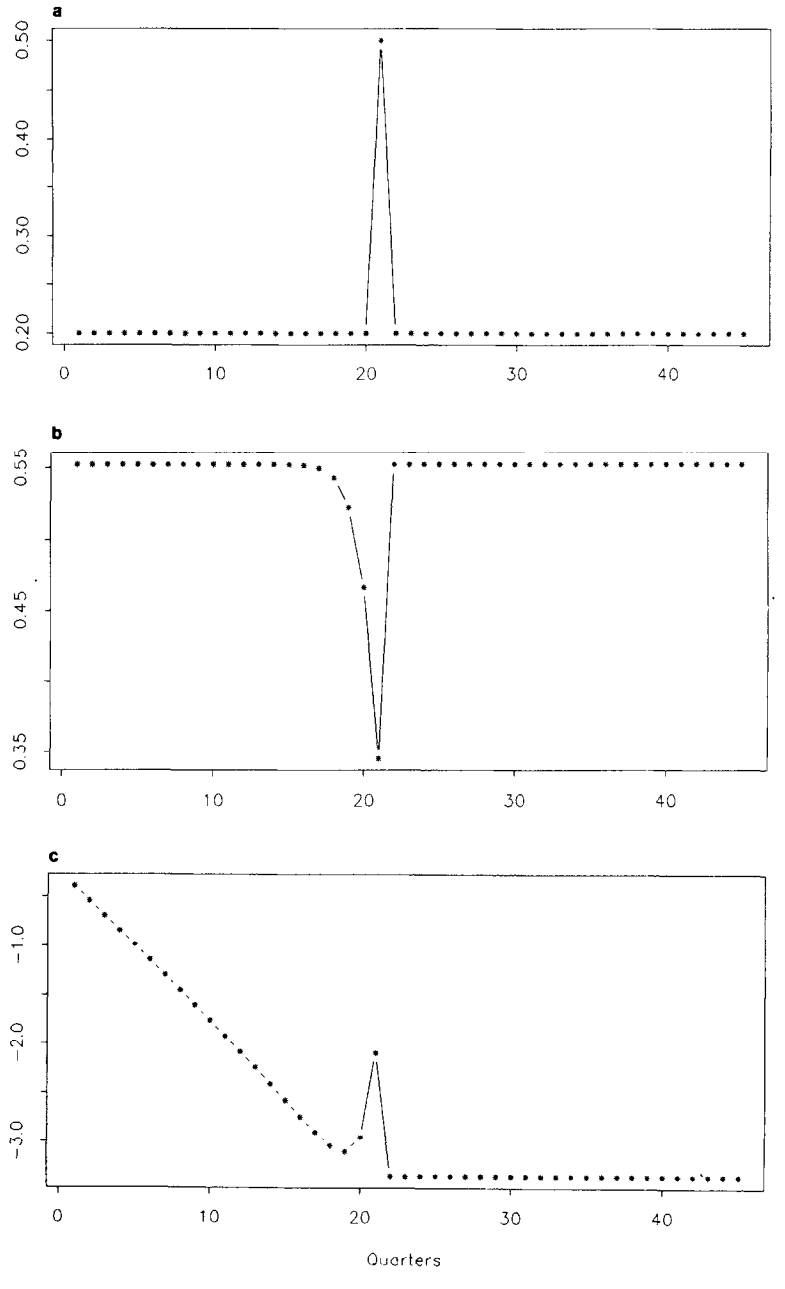

FIG. 1. The dynamics of Ramsey equilibrium when there is a temporary increase in government expenditure/output ratio: (a) the government expenditure/output ratio; (b) the consumption/output ratio; (c) the public debt issue/output ratio; (d) the gross investment/output ratio; (e) the labor income tax rate; (f) the employment rate.

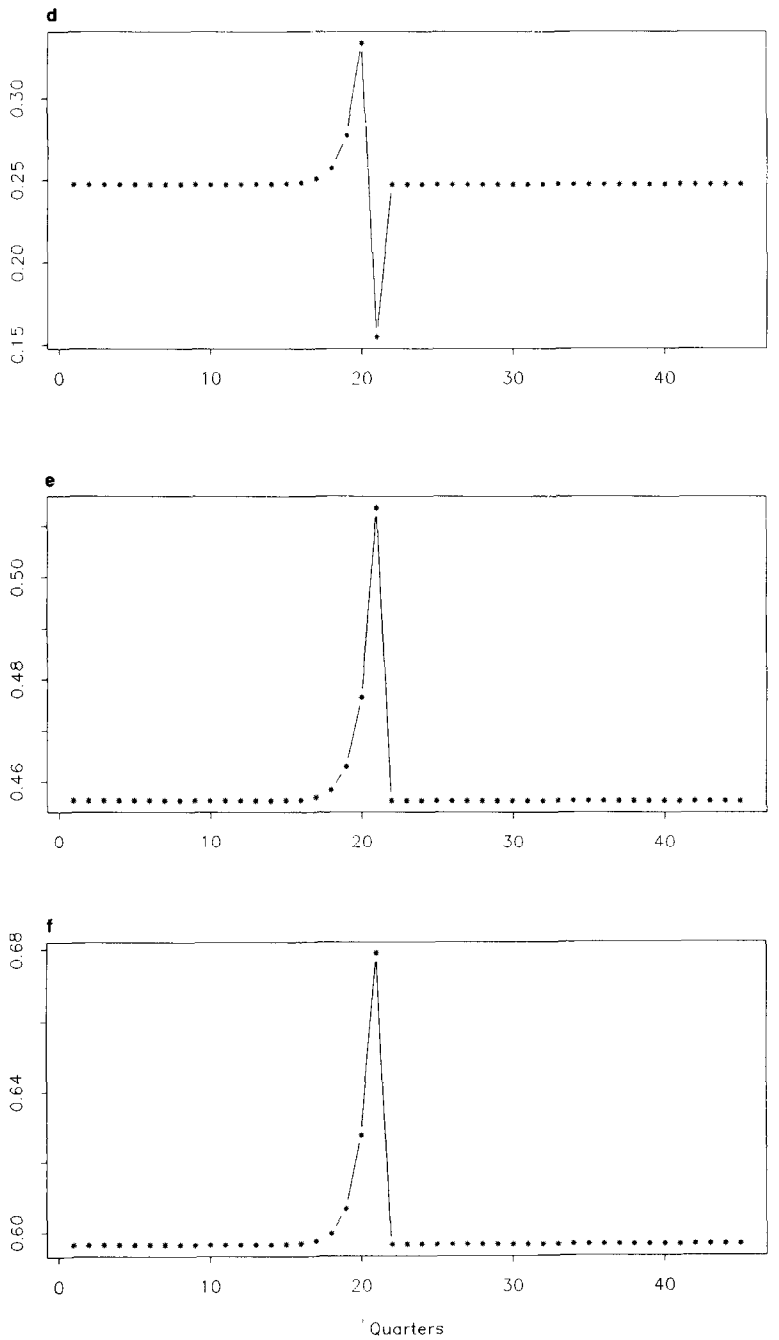

FIG. 1—*Continued*

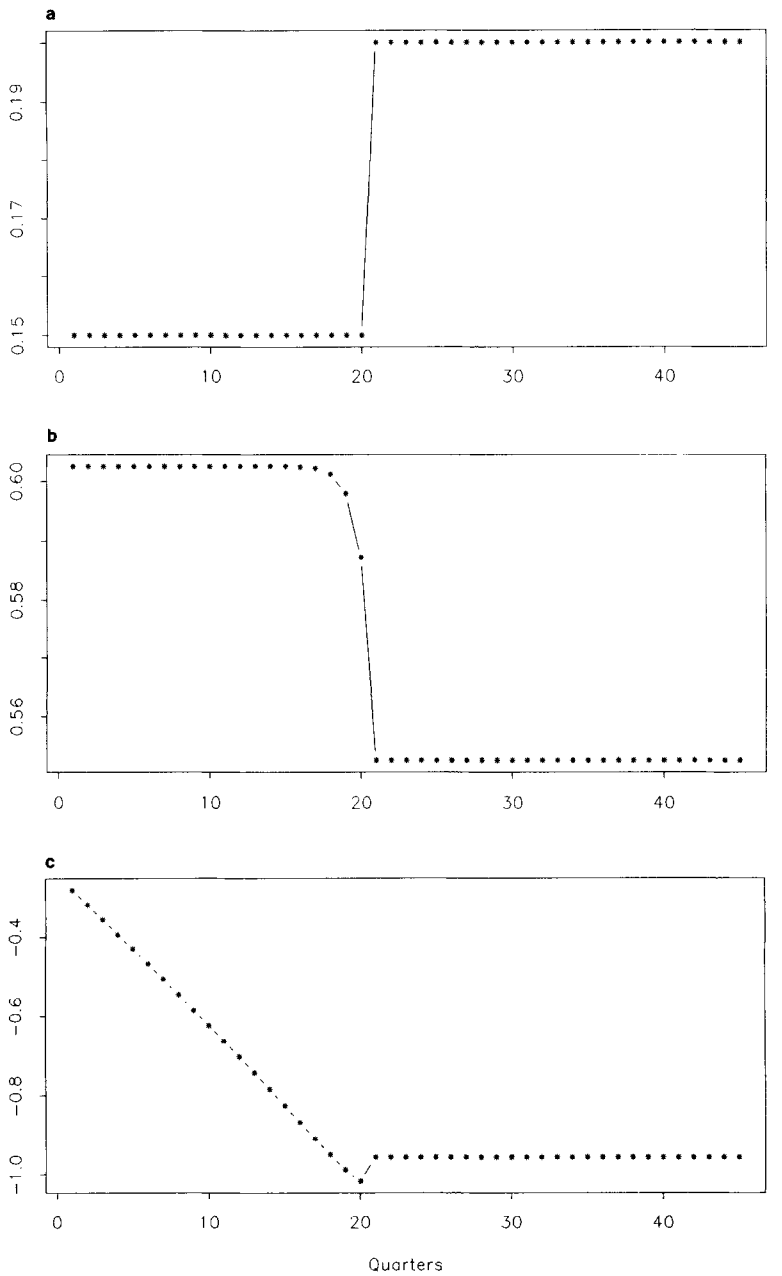

FIG. 2. The dynamics of Ramsey equilibrium when there is a permanent increase in government expenditure/output ratio: (a) the government expenditure/output ratio; (b) the consumption/output ratio; (c) the public debt issue/output ratio; (d) the gross investment/output ratio; (e) the labor income tax rate; (f) the employment rate.

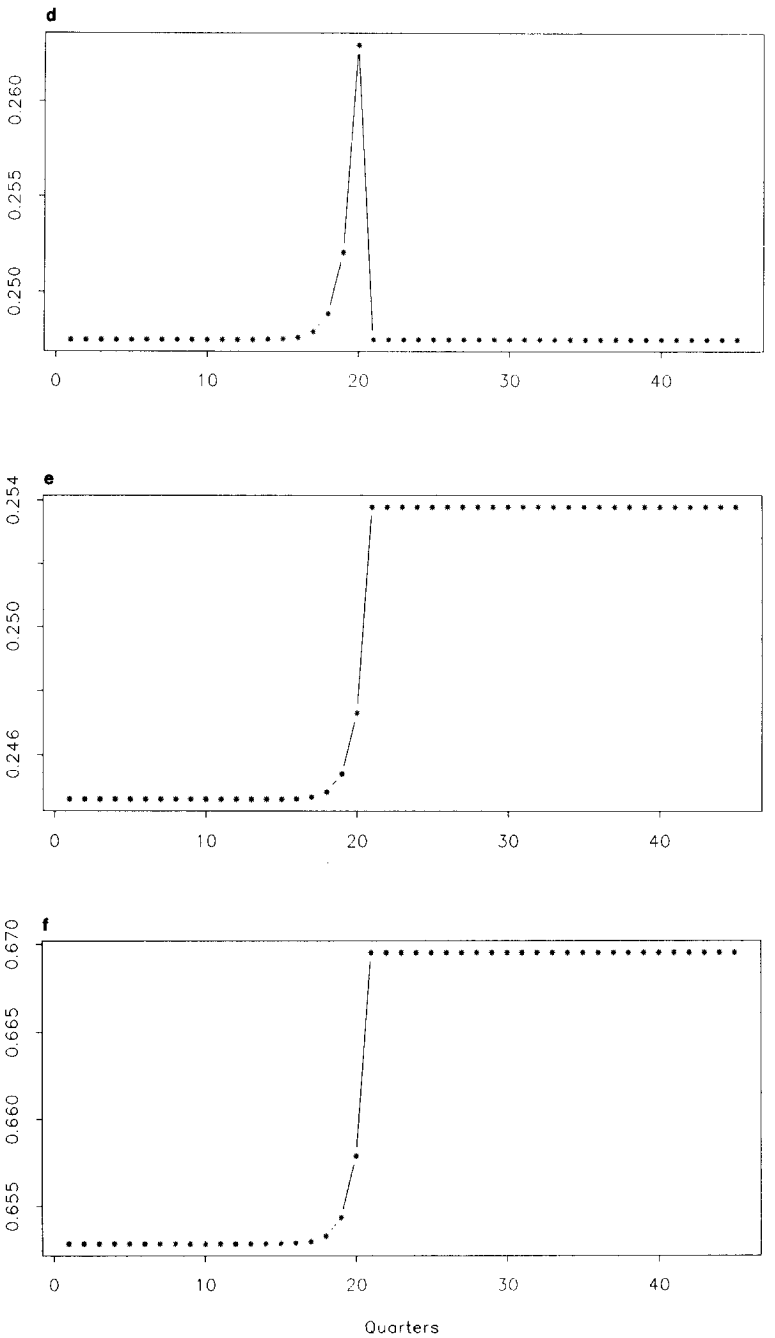

FIG. 2—*Continued*

6. Conclusion

This paper theoretically investigated the implications of optimal precommited fiscal policy in the framework of dynamic Ramsey taxation. While this framework is a good benchmark case for the study of optimal fiscal policy, many important issues about optimal fiscal policy are left unaddressed. In this section I point out some of those issues that I think are important and need to be addressed in future researches.

First, this paper assumes that the economy consists of identical agents and the analyses have concentrated on studying the optimal fiscal policy that would achieve economic efficiency. But many times government is concerned about the equity as well as efficiency in the economy when it designs its fiscal policy. For example, the tax on capital income has usually been justified as a means of transferring wealth among members of the society. It would be extremely interesting to see how the implications of optimal fiscal policy change when the government is concerned about distributional effects of its policy. In [33], competitive equilibria with capital and labor income taxes are studied in a economy with heterogenous agents. It is shown there that, for any given quantity distribution the government wants to achieve through transfers and capital and labor income taxation, there are always an infinite number of different combinations of capital income tax rates and public debt payments that can implement the same quantity distribution. In other words, the indeterminacy of fiscal policy continues to exist even in economies with heterogenous agents where fiscal policy has distributional effects.

Second, this paper assumes that government expenditures play no specific roles in the economy and are taken as exogenously given by both the government and the household. In Zhu [33], the structure of optimal capital and labor income taxation is studied when the government expenditures are public goods which enter into the consumer's utility function as well as the private production function. It is shown there that the structure of optimal capital and labor income taxation depends on how one models the role of government expenditures in private production, but, in general, does not depend on the endogeneity of government expenditures. For example, if government expenditures are pure public goods that enter into the private production function and that the production is constant return to scale in private inputs, then most of the results in this paper about the structure of optimal capital and labor income taxation carry through even if the government *chooses* the levels of government expenditures endogenously. On the other hand, if the production function is decreasing return in private inputs but constant return in private and public inputs, then the structure of optimal capital and labor income taxation will be quite different from the structure that is found in this paper

even if the levels of government expenditures are taken as exogenously given.

Third, this paper assumes that the government has the ability to precommit itself. It is well known that when the government is free to change its policy in every period, it has and incentive to levy 100% tax on capital income in every period. An important question is, then, what would make the government follow the optimal fiscal policy that is set in the intial period? In some very simplified models Chari and Kehoe [11] and Stockey [30] show that the optimal precommited policy may sustain as a time consistent policy if the consumer's discount rate is large. It would be very interesting to see whether their results can be extended to the model that is considered in this paper. Another interesting question is whether government can resolve the time-inconsistency problem by using investment credit. That is, can government use investment credit to bind future governments to follow the optimal fiscal policy that is set in the initial period? All these questions are interesting topics for future researches.

5. Appendix

Proof of Proposition 1. By definition, an allocation is implementable if and only if it satisfies conditions (2.1) and (3.3) through (3.9) for some price sequence $(w_t, r_t, p_{t+1})_{t\geq 0}$ and a fiscal policy $(\theta_t, \tau_t, b_{t+1})_{t\geq 0}$ that satisfies (3.1). Let $(c_t, n_t, k_{t+1})_{t\geq 0}$ be such an allocation. Since it has been assumed that there are no shocks in period zero, the constraint (3.1) becomes (4.1) when $t=0$. For $t\geq 0$, (4.2) is obtained by substituting (3.4) and (3.5) into (3.1) and (4.6) is obtained by substituting (3.4) into (3.8). Finally, by substituting (3.3) through (3.5), and (3.9) into the budget constraint (3.6), we have

$$u_1(c_t, 1-n_{t+1})(b_t+\bar{R}_t k_t) = u_1(c_t, 1-n_t)\, c_t - u_2(c_t, 1-n_t)\, n_t + \beta E_t\, u_1(c_{t+1}, 1-n_{t+1})(b_{t+1}+\bar{R}_{t+1}k_{t+1}).$$

Then, (4.4) is obtained by solving the above difference equation forward using the transversality conditions (3.7) and (3.8). Up to now, we have proved that any implementable allocation will satisfy (4.1) through (4.6). Next, assume that there is an allocation $(c_t, n_t, k_{t+1})_{t\geq 0}$ that satisfies conditions (4.1) through (4.6). Then, for every $t\geq 0$, let $r_t = F_1(k_t, n_t; z_t)$, $w_t = F_2(k_t, n_t; z_t)$, and $\theta_t = 1 - w_t^{-1}[u_l(c_t, 1-n_t)/u_c(c_t, 1-n_t)]$; let τ_t be any random variable that satisfies (3.1); and let

$$b_t = -\bar{R}_t k_t + \frac{1}{u_1}(c_t, 1-n_t) \times \sum_{j=0}^{\infty} \beta^j E_t[u_1(c_{t+j}, 1-n_{t+j})\, c_{t+j} - u_2(c_{t+1}, 1-n_{t+j})\, n_{t+j}].$$

It is straightforward to verify that the allocation with market prices, the tax rates, and public debt payments defined above satisfies conditions (3.3) through (3.9). Therefore, it is an implementable competitive allocation.

Q.E.D.

Proof of Proposition 2. We prove the proposition by applying a generalized Kuhn–Tucker theorem of Bazaraa and Goode [6]. In order to do this, we employ the following definitions and notations:

(i) *Choice Space.* Let $\mathbf{S} \subset \mathbf{R}^m$ be the state space for the state variable s_t, $t > 0$. For given s_0, let P_t be the joint probability distribution of $(s_1, ..., s_t)$ on $\mathbf{B}(\mathbf{S}^t)$, and let $\mathbf{X}_t = L_2(\mathbf{S}^t, \mathbf{B}(\mathbf{S}^t)\, P_t)$ for every $t > 0$. Let $\mathbf{X} = \{\{x_t\}_{t \geq 0} \in \prod_{t=0}^{\infty} \mathbf{X}_t : \sum_{t=0}^{\infty} \gamma^t \|x_t\|_2 < +\infty\}$, where $\gamma = .5\beta(1-\delta)$ or $.5\beta$ depending on whether $\delta < 1$ or $\delta = 1$, and $\|\cdot\|_2$ denotes the standard norm in L_2 spaces. Let $c = \{c_t\}_{t \geq 0}$, $n = \{n_t\}_{t \geq 0}$, and $k = \{k_{t+1}\}_{t \geq 0}$. Now, the choice space of the Ramsey problem can be defined as

$$\mathbf{D} = \{(\tau_0, c, n, k) : \tau_0 \in \mathbf{R}_1, (c, n, k) \in \mathbf{X}^3\}.$$

Define a norm $\|\cdot\|_*$ on $\mathbf{D}$ as follows:

$$\|(\tau_0, c, n, k)\|_* = |\tau_0| + \sum_{t=0}^{\infty} \gamma^t (\|c_t\|_2 + \|n_t\|_2 + \|k_{t+1}\|_2).$$

Then $\mathbf{D}$ is a Banach space.

Since the production function is diminishing returns to scale in k_t, there are upper bounds $\bar{k}$ and $\bar{y}$ such that $c_t \leqslant \bar{y}$ and $k_t \leqslant \bar{k}$ for all $t \geqslant 0$ if (c, n, k) is a competitive allocation. Therefore, for any competitive allocation (c, n, k) and any τ_0, $(\tau_0, c, n, k) \in \mathbf{D}$.

(ii) *Feasible Set.* Let $\mathbf{Q} = \{(c, n, k) \in \mathbf{X}^3 : \lim_{t \to \infty} E_0 \beta^t u_c(c_t, 1 - n_t) k_{t+1} = 0\}$. We show that $\mathbf{Q}$ is an open set. Let $(c, n, k) \in \mathbf{Q}$. Since the function $v(c_t, n_t, k_{t+1}) = u_1(c_t, 1 - n_t) k_{t+1}$ is continuous in the region $\{(c_t, n_t, k_{t+1}) : c_t > 0,\ 0 < n_t < 1,\ k_{t+1} > 0\}$, there is an $\varepsilon > 0$ such that for every $t > 0$,

$$\begin{aligned} &\|c_t' - c_t\|_2 + \|n_t' - n_t\|_2 + \|k_{t+1}' - k_{t+1}'\|_2 < \varepsilon \\ &\quad \Rightarrow E_0 v(c_t', n_t', k_{t+1}') - E_0 v(c_t, n_t, k_{t+1})| < 1. \end{aligned} \tag{A.1}$$

Thus, for any (c', n', k') which satisfies $\|\underline{c}', \underline{n}', \underline{k}') - (c, n, k)\|_* < \varepsilon$,

$$\begin{aligned} &E_0 \beta^t v(c_t', n_t', k_{t+1}') \\ &\quad = \beta^t E_0 v(c_t', n_t', k_{t+1}') - \beta^t E_0 v(c_t, n_t, k_{t+1}) + \beta^t E_0 v(c_t, n_t, k_{t+1}) \\ &\quad \leqslant \beta^t |E_0 v(c_t', n_t', k_{t+1}') - E_0 v(c_t, n_t, k_{t+1})| + \beta^t E_0 v(c_t, n_t, k_{t+1}) \\ &\quad < \beta^t + \beta^t E_0 v(c_t, n_t, k_{t+1}); \end{aligned} \tag{A.2}$$

therefore,

$$\lim_{t\to\infty} E_0\beta^t v(c'_t, n'_t, k'_{t+1}) < \lim_{t\to\infty} \beta^t + \lim_{t\to\infty} E_0\beta^t v(c_t, n_t, k_{t+1}) = 0.$$

That is,

$$\|(c', n', k') - (c, n, k)\| < \varepsilon \Rightarrow \lim_{t\to\infty} E_0\beta^t u_1(c_t, 1-n_t)\, k_{t+1} = 0. \quad \text{(A.3)}$$

Equation (A.3) is true for any $(c, n, k) \in \mathbf{Q}$; hence, $\mathbf{Q}$ is an open set.

(iii) *Operators.* $\mathbf{U}: \mathbf{D} \mapsto \mathbf{R}^1$,

$$\mathbf{U}(\tau_0, c, n, k) = E_0 \sum_{t=0}^{\infty} \beta u(c_t, 1-n_t).$$

$\mathbf{G}^{(1)}: \mathbf{D} \mapsto \mathbf{R}^1$,

$$\mathbf{G}^{(1)}(\tau_0, c, n, k) = 1 - \tau_0.$$

$\mathbf{G}^{(2)}: \mathbf{D} \mapsto \mathbf{X}$. If (τ_0, c, n, k) is such that

$$\sum_{t=0}^{\infty} \gamma^t \|u_1(c_t, 1-n_t) - \beta(1-\delta)\, E_t u_1(c_{t+1}, 1-n_{t+1})\|_2 < \infty,$$

then

$$[\mathbf{G}^{(2)}(\tau_0, c, n, k)]_t = u_1(c_t, 1-n_t) - \beta(1-\delta)\, E_t u_1(c_{t+1}, 1-n_{t+1}).$$

Otherwise,

$$[\mathbf{G}^{(2)}(\tau_0, c, n, k)]_t = 0.$$

$\mathbf{G}^{(3)}: \mathbf{D} \mapsto \mathbf{X}$. If (τ_0, c, n, k) is such that

$$\sum_{t=0}^{\infty} \gamma^t \|F(k_t, n_t; z_t) + (1-\delta)\, k_t - k_{t+1} - c_t - g_t\|_2 < \infty,$$

then

$$[\mathbf{G}^{(3)}(\tau_0, c, n, k)]_t = F(k_t, n_t; z_t) + (1-\delta)\, k_t - k_{t+1} - c_t - g_t.$$

Otherwise,

$$[\mathbf{G}^{(3)}(\tau_0, c, n, k)]_t = 0.$$

$\mathbf{G}: \mathbf{D} \mapsto \mathbf{R}^1 \times \mathbf{X}^2$,

$$\mathbf{G} = (\mathbf{G}^{(1)}, \mathbf{G}^{(2)}, \mathbf{G}^{(3)}).$$

$\mathbf{H}: \mathbf{D} \mapsto \mathbf{R}^1$,

$$\mathbf{H}(\tau_0, c, n, k) = u_1(c_0, 1-n_0)[b_0 + \bar{R}_0 k_0] - E_0 \sum_{t=0}^{\infty} \beta^t [u_1(c_t, 1-n_t)\, c_t - u_2(c_t, 1-n_t)\, n_t].$$

(iv) *Ramsey problem.* Now the Ramsey problem can be stated as

$$\begin{aligned} &\text{maximize} && \mathbf{U}(d), \\ &\text{subject to} && d \in \mathbf{R}^1 \times \mathbf{Q} \subset \mathbf{D},\ \mathbf{G}(d) \geqslant 0,\ H(d) = 0. \end{aligned}$$

Applying Theorem 2.3 of Bazarra and Goode [6] to the above problem, we have Proposition 2. Q.E.D.

Proof of Proposition 3. By the definitions of the distance function d, a_t^c, and a_t^n, we have, for $t \geqslant 0$,

$$a_t^c = \frac{\beta^t \pi(s^t)\, u_1(c_t/d,\ 1 - n_t/d)}{H},$$

$$a_t^n = \frac{\beta^t \pi(s^t)\, u_2(c_t/d,\ 1 - n_t/d)}{H},$$

where

$$H = E_0 \sum_{i=0}^{\infty} \beta^i [u_1(c_i/d,\ 1 - n_i/d)\, c_i/d - u_2(c_i/d,\ 1 - n_i/d)\, n_i/d].$$

Thus, for $t > 0$,

$$\frac{\partial \log(a_t^c)}{\partial c_0} = -\frac{\partial \log(H)}{\partial c_0} - \frac{u_{11}(c_t/d,\ 1 - n_t/d)\, c_t/d - u_{12}(c_t/d,\ 1 - n_t/d)\, n_t/d}{u_1(c_t/d,\ 1 - n_t/d)} \frac{1}{d} \frac{\partial d}{\partial c_0},$$

$$\frac{\partial \log(a_t^n)}{\partial c_0} = -\frac{\partial \log(H)}{\partial c_0} - \frac{u_{12}(c_t/d,\ 1 - n_t/d)\, c_t/d - u_{22}(c_t/d,\ 1 - n_t/d)\, n_t/d}{u_2(c_t/d,\ 1 - n_t/d)} \frac{1}{d} \frac{\partial d}{\partial c_0},$$

and for $t = 0$,

$$\frac{\partial \log(a_0^c)}{\partial c_0} = -\frac{\partial \log(H)}{\partial c_0} - \frac{u_{11}(c_0/d,\ 1 - n_0/d)\, c_0/d - u_{12}(c_0/d,\ 1 - n_0/d)\, n_0/d}{u_1(c_0/d,\ 1 - n_0/d)} \frac{1}{d} \frac{\partial d}{\partial c_0} + \frac{u_{11}(c_0/d,\ 1 - n_0/d)}{u_1(c_0/d,\ 1 - n_0/d)}$$

$$\frac{\partial \log(a_0^n)}{\partial c_0}$$

$$= -\frac{\partial \log(H)}{\partial c_0} - \frac{u_{12}(c_0/d, 1-n_0/d)\, c_0/d - u_{22}(c_0/d, 1-n_0/d)\, n_0/d}{u_2(c_0/d, 1-n_0/d)} \frac{1}{d} \frac{\partial d}{\partial c_0}$$

$$+ \frac{u_{12}(c_0/d, 1-n_0/d)}{u_2(c_0/d, 1-n_0/d)}.$$

For a sequence $\{c_t, n_t\}_{t \geq 0}$ that satisfies the implementability constraint (4.4) and that $\bar{u} = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$, we have $d=1$ and $H = u_1(c_0, 1-n_0)[b_0 + \bar{R}_0 k_0]$. By substituting them into the above equations we have (4.18) and (4.19) of Proposition 3. Q.E.D.

Proof of Proposition 4. (i) Since v is homogenous in c and n, there is a constant α such that

$$v_1(c, n)c + v_2(c, n)n = \alpha v(c, n).$$

Taking partial derivatives with respect to c on both sides of the equation, we have

$$v_{11}(c, n)c + v_{12}(v, n) = \alpha v_1(c, n),$$

which implies that $\Lambda_t^c = \alpha$. $\bar{\tau}_{t+1} = 0$ then follows from Eq. (4.17).

(ii) Similar to the proof of (i).

Q.E.D.

Proof of Proposition 5. In order to prove Proposition 5, I first prove three lemmas. From Proposition 2, the following first order conditions hold for the Ramsey allocation:

$$\lambda_t = \beta^t \pi_t(s^t)\, u_1(c_t, 1-n_t) \left[1 + \Phi \Lambda_t^c + \frac{u_{11}(c_t, 1-n_t)}{u_1(c_t, 1-n_t)} (\mu_t - (1-\delta)\, \mu_{t-1}) \right], \tag{A.4}$$

$$\lambda_t = \frac{\beta^t \pi_t(s^t)\, u_2(c_t, 1-n_t)}{F_n(k_t, n_t, z_t)} \left[1 + \Phi \Lambda_t^n + \frac{u_{12}(c_t, T-n_t)}{u_2(c_t, 1-n_t)} (\mu_t - (1-\delta)\, \mu_{t-1}) \right], \tag{A.5}$$

$$\lambda_t = \int \lambda_{t+1} R_{t+1}\, ds_{t+1}. \tag{A.6}$$

Let H_t denote $\lambda_t / \beta^t \pi_t(s^t)\, u_1(c_t, 1-n_t)$. Then, from (A.4)–(A.6),

$$u_1(c_t, 1-n_t)\, H_t = \beta E_t u_c(c_{t+1}, 1-n_{t+1})\, H_{t+1} R_{t+1}, \tag{A.7}$$

and

$$H_t = \frac{u_2(c_t, 1-n_t)u_{11}(c_t, 1-n_t)(1+\Phi\Lambda_t^n) - u_1(c_t, 1-n_t)u_{12}(c_t, 1-n_t)(1+\Phi\Lambda_t^c)}{u_{11}(c_t, 1-n_t)F_2(k_t, n_t; z_t) - u_{12}(c_t, 1-n_t)}. \tag{A.8}$$

LEMMA 1. *In the stationary equilibrium,*

$$\bar{\tau}_{t+1} \gtreqless 0 \Leftrightarrow H_t \gtreqless \frac{E_t u_1(c_{t+1}, 1-n_{t+1})\, R_{t+1} H_{t+1}}{E_t u_1(c_{t+1}, 1-n_{t+1})\, R_{t+1}}, \tag{A.9}$$

for every $t \geqslant 0$.

Proof. Note that $\bar{\tau}_{t+1} \gtreqless 0$ if and only if $E_t u_1(c_{t+1}, 1-n_{t+1})\, F_1(k_{t+1}, n_{t+1}; z_{t+1})\, \tau_{t+1} \gtreqless 0$. Equation (A.9) then follows from comparing Eq. (A.7) to the competitive equilibrium condition (3.4). Q.E.D.

Since it has been assumed that c_t and n_t are time-invariant, continuous functions of k_t and s_t, and since H_t, $u_c(c_t, 1-n_t)\, R_t$ and $\bar{\tau}_{t+1}$ are all time-invariant functions of c_t, n_t, k_t and s_t, one can write them as follows:

$$H_t = H(k_t, s_t), \qquad u_1(c_t, c_t, 1-n_t)\, R_t = p(k_t, s_t), \qquad \bar{\tau}_{t+1} = \bar{\tau}(k_t, s_t),$$

where $H(\cdot, \cdot)$, $p(\cdot, \cdot)$ and $\bar{\tau}(\cdot, \cdot)$ are continuous functions on $[0, \bar{k}] \times \mathbf{S}$.

Let $\mathscr{F}$ be the space of continuous functions on $[0, \bar{k}] \times \mathbf{S}$ and let T be an operator on $\mathscr{F}$,

$$Tf(k, s) = \frac{\int f(k', s')\, p(k', s')\, \pi(s', s)\, ds'}{\int p(k', s')\, \pi(s', s)\, ds'}, \tag{A.10}$$

where $k' = \phi(k, s)$. Then another way to state Lemma 1 is that

$$\bar{\tau}(k, s) \gtreqless 0 \Leftrightarrow H(k, s) - TH(k, s) \gtreqless 0.$$

Thus, the sign of the expected optimal capital income tax rate is totally determined by the sign of the function $H - TH$. For the later, we have

LEMMA 2. *If Assumption* 1 *and* 2 *are maintained, then,* $P^\infty\{(k, s)$: $H(k, s) - TH(k, s) - TH(k, s) = 0\} = 1$ *if and only if there is a constant* h^* *such that* $P_\infty\{(k, s)$: $H(k, s) = h^*\} = 1$.

Proof. First define some notations. For any $(k_0, s_0) \in [0, \bar{k}] \times \mathbf{S}$, let $k_t = \phi(k_{t-1}, s_{t-1})$, $s_t \in \mathbf{S}$, $s_0^t = (s_1, ..., s_t)$, $k_0^t = (k_1, ..., k_t)$, and $\pi^t(s_0^t, s_0) = \prod_{j=1}^t \pi(s_j, s_{j-1})$, $t \geqslant 1$. Then, for any $A \in B([0, \bar{k}] \times B(\mathbf{S})$, by definition,

$$P^\infty\{A\} = \text{Prob}\{(k_t, s_t) \in A\} = \int \text{Prob}\{(k_t, s_t) \in A \mid k_0, s_0\}\, P^\infty(dk_0, ds_0),$$

$$\text{Prob}\{(k_t, s_t) \in A \mid k_0, s_0\} = \int_{(k_t, s_t) \in A} \pi^t(s_0^t, s_0)\, ds_0^t.$$

Let

$$\bar{p}^t(k_0^t, s_0^t) = \prod_{j=1}^{t} \frac{p(k_t, s_t)}{\int p(k_j, s_j)\, \pi(s_j, s_{j-1})\, ds_j}.$$

Then

$$\int \bar{p}^t(k_0^t, s_0^t)\, \pi^t(s_0^t, s_0)\, ds_0^t = 1,$$

and, by the definition of T,

$$T^t H(k_0, s_0) = \int H(k_t, s_t)\, \bar{p}^t(k_0^t, s_0^t)\, \pi^t(s_0^t, s_0)\, ds_0^t.$$

Suppose $P^\infty\{(k, s): H(k, s) = TH(k, s)\} = 1$. The lemma is proved in several steps.

(i) Step One: $P^\infty\{(k_0, s_0): H(k_0, s_0) = T^t H(k_0, s_0),\ t > 0\} = 1$. Let $A_t = \{(k_0, s_0): \text{Prob}\{(k_t, s_t) = TH(k_t, s_t) \mid k_0, s_0 \hat{\partial} = 1\}$, for all $t \geqslant 0$. Because

$$1 = P^\infty\{(k_t, s_t): H(k_t, s_t) = TH(k_t, s_t)\}$$

$$= \int \text{Prob}\{(k_t, s_t): H(k_t, s_t) = TH(k_t, s_t) \mid k_0, s_0\}\, P^\infty(dk_0, ds_0),$$

so $P^\infty(A_t) = 1$. If $A = \bigcap_{t=0}^{\infty} A_t$, then $P^\infty\{A\} = 1$. But for $(k_0, s_0) \in A_t$, we have that $H(k_{t-1}, s_{t-1}) = TH(k_{t-1}, s_{t-1})$. Thus, for $(k_0, s_0) \in A$, we have that $H(k_0, s_0) = T^t(k_0, s_0)$ or $P^\infty\{(k_0, s_0): H(k_0, s_0) = T^t H(k_0, s_0),\ t > 0\} = 1$.

(ii) Step Two: There is a constant h^* such that $P^\infty\{(k_0, s_0): \text{Prob}\{H(k_t, s_t) \leqslant h^* \mid k_0, s_0\} = 1,\ t > 0\} = 1$. Let $h^* = \min\{h: P^\infty\{H(k, s) \leqslant h\} = 1\}$ and let $B_t = \{(k_0, s_0): \text{Prob}\{H(k_t, s_t) \leqslant h^* \mid k_0, s_0\} = 1\}$. Because

$$1 = P^\infty\{H(k_t, s_t) \leqslant h^*\}$$

$$= \int \text{Prob}\{H(k_t, s_t) \leqslant h^* \mid k_0, s_0\}\, P^\infty(dk_0, ds_0),$$

so, $P^\infty\{B_t\} = 1$. If $B = \bigcap_{t=0}^{\infty} B_t$, then $P^\infty\{B\} = 1$. That is, $P^\infty\{(k_0, s_0): \text{Prob}\{H(k_t, s_t) \leqslant h^* \mid k_0, s_0\} = 1,\ t > 0\} = 1$.

(iii) Step Three: $P^\infty\{(k, s): H(k, s) = h^*\} = 1$. From (i) and (ii), we have that $P^\infty\{A\} = P^\infty\{B\} = 1$. Thus, $P^\infty\{A \cap B\} = 1$. Since the functions

$H(\cdot,\cdot)$ and $T^tH(\cdot,\cdot)$ are all continuous, A is a closed set. By Assumption 1, however, the function $\text{Prob}\{H(k_t, s_0)\}$ is also a continuous function of k_0 and s_0, thus B is also a closed set, and so is $A \cap B$. By the definition of h^*, there is at least one $(k_0, s_0) \in A \cap B$ such that $H(k_0, s_0) \geq h^*$. (Otherwise, $H(k_0, s_0) < h^*$ for all $(k_0, s_0) \in A \cap B$. But since $H(\cdot,\cdot)$ is a continuous function on $A \cap B$ which is a closed subset of the compact space $[0, \bar{k}] \times \mathbf{S}$, $H(\cdot,\cdot)$ has a maximum h which is less than h^*. This contradicts the definition of h^*.)

Let $(k_0, s_0) \in A \cap B$ such that $H(k_0, s_0) \geq h^*$. Since $(k_0, s_0) \in A$, for every $t > 0$,

$$H(k_0, s_0) = T^tH(k_0, s_0) = \int H(k_t, s_t)\, \bar{p}^t(k_0^t, s_0^t)\, \pi^t(s_0^t, s_0)\, ds_0^t. \quad \text{(A.11)}$$

But $(k_0, s_0) \in B$, for every $t > 0$,

$$\text{Prob}\{H(k_t, s_t) \leq h^* \mid k_0, s_0\} = 1. \quad \text{(A.12)}$$

Combining (A.11) and (A.12), then, one has that

$$\text{Prob}\{H(k_t, s_t = h^* \mid k_0, s_0\} = 1, \qquad \text{for all} \quad t > 0. \quad \text{(A.13)}$$

Finally, since (k_t, s_t) is ergodic,

$$P_\infty\{H(k, s) = h^*\} = \lim_{t \to \infty} \text{Prob}\{H(k_t, s_t) = h^* \mid k_0, s_0\} = 1. \quad \text{Q.E.D.}$$

LEMMA 3. *If Assumptions* 1 *and* 2 *are maintained, then the following three statements are equivalent*:

$$\text{(i)} \quad \text{Prob}\{(k, s): H(k, s) - TH(k, s) \geq 0\} = 1$$

$$\text{(ii)} \quad \text{Prob}\{(k, s): H(k, s) - TH(k, s) \leq 0\} = 1$$

$$\text{(iii)} \quad \text{Prob}\{(k, s): H(k, s) - TH(k, s) = 0\} = 1.$$

Proof. The proof is very similar to the proof of Lemma 2. I sketch the proof of the equivalence of (i) and (iii).

Suppose $\text{Prob}\{(k, s): H(k, s) - TH(k, s) \geq 0\} = 1$. Let $h^* = \max\{h: P^\infty\{H(k, s) \geq h\} = 1\}$, $A_t = \{(k, s): H(k_0, s_0) \geq T^tH(k_0, s_0)\}$, $B_t = \{(k_0, s_0): \text{Prob}\{H(k_t, s_t) \geq h^*\} = 1\}$, $A = \bigcap_{t=0}^{\infty} A_t$, and $B = \bigcap_{t=0}^{\infty} B_t$, then $A \cap B$ is a closed set and $P^\infty\{A \cap B\} = 1$.

By the definition of h^*, then, there is a $(k_0, s_0) \in A \cap B$ such that $H(k_0, s_0) \leq h^*$, which implies that $\text{Prob}\{H(k_t, s_t) = h^*\} = 1$. Using the ergodicity of (k_t, s_t), then, we have

$$P^\infty\{H(k, s) = h^*\} = \lim_{t \to \infty} \text{Prob}\{H(k_t, s_t) = h^* \mid k_0, s_0\} = 1.$$

In other words, (i) and (iii) are equivalent. The proof of the equivalence of (ii) and (iii) follows the same logic. Q.E.D.

Now I prove Proposition 5. Assume that either $\text{Prob}\{\bar{\tau}_{t+1} \geqslant 0\} = 1$ or $\text{Prob}\{\bar{\tau}_{t+1} \leqslant 0\} = 1$. From Lemma 1, either $\text{Prob}\{(k, s)\colon H(k, s) - TH(k, s) \leqslant 0\} = 1$ or $\text{Prob}\{(k, s)\colon H(k, s) - TH(k, s) \leqslant 0\} = 1$, which in turn implies, from Lemma 3, that $\text{Prob}\{(k, s)\colon H(k, s) - TH(k, s) = 0\} = 1$. But from Lemma 2, $\text{Prob}\{(k, s)\colon H(k, s) - TH(k, s) = 0\} = 1$ implies $\text{Prob}\{\bar{\tau}_{t+1} = 0\} = 1$. Thus we have proved (i) of Proposition 5. (ii) of the proposition follows from the fact that, when $\text{Prob}\{\bar{\tau}_{t+1} = 0\} = 1$, the boundary condition (4.2) is not binding so that $H = 1 + \Phi A^c$. (iii) of the proposition follows from (ii) and Proposition 3. Q.E.D.

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