Soft Budget Constraint and Inflation Cycles: A Positive Model of the Macro Dynamics in China during Transition

Appendix[1]

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[1] This appendix contains proofs of propositions.

Proof of Proposition 4:

There exists an equlibrium with positive price levels if and only if the inflation rate given in equation (9) is positive, which is equivalent to the condition

$$\rho_t < \frac{Q_l A_g + (1-Q_l)A_b + \gamma Q_l (A_g - R)}{[\alpha\gamma(A_g - R) - A_g]Q_t} \equiv \rho^*(\alpha, Q_t),$$

or

$$\lambda_t < \frac{Q_l A_g + (1-Q_l)A_b + \gamma Q_l(A_g - R)}{(2-Q_l)[\alpha\gamma(A_g - R) - A_g]Q_t + Q_l A_g + (1-Q_l)A_b + \gamma Q_l(A_g - R)} \equiv \lambda^*(\alpha, Q_t).$$

Apparently, for $\alpha > A_g(A_g - R)^{-1}\gamma^{-1}$, $\lambda^*(\alpha, Q_t) \in (0,1)$ and it is a decreasing function in both α and Q_t. It is also easy to see from (9) that $\lim_{\rho_t \to \rho^*(\alpha,Q_t)} p_{t+1}/p_t = +\infty$. Finally, note that $\lim_{\lambda_t \to \lambda^*(\alpha,Q_t)} \rho_t = \rho^*(\alpha, Q_t)$.Q.E.D.

Proof of Proposition 5:

For any τ, define $\widehat{\rho}(\tau, \alpha, Q_t)$ as follows:

$$\widehat{\rho}(\tau, \alpha, Q_t) = \frac{\tau[Q_l A_g + (1-Q_l)A_b + \gamma Q_l(A_g - R)] - \gamma(\gamma-1)^{-1}(2-Q_l)}{\gamma(\gamma-1)^{-1} + \tau[\alpha\gamma(A_g - R) - A_g]Q_t}.$$

From the definition of $\underline{\tau}$, $\tau > \underline{\tau}$ implies $\widehat{\rho}(\tau, \alpha, Q_t) > 0$. One can easily verify from (9) that letting $\rho_t = \widehat{\rho}(\tau, \alpha, Q_t)$ yields a constant inflation rate τ. From (9) and (10), we also have

$$\frac{Y_{t+1}}{Y_t} = \frac{A_g Q_t \rho_t + [Q_l A_g + (1-Q_l)A_b]}{[A_g - \alpha\gamma(A_g - R)]Q_t\rho_t + [Q_l A_g + (1-Q_l)A_b] + Q_l\gamma(A_g - R)}\tau^{-1},$$

which is increasing in $Q_t\rho_t$. But from the definition of $\widehat{\rho}(\tau, \alpha, Q_t)$ we can see that $Q_t\widehat{\rho}(\tau, \alpha, Q_t)$ is increasing in Q_t. Thus, when $\rho_t = \widehat{\rho}(\tau, \alpha, Q_t)$, the output growth rate is increasing in Q_t. To complet the proof, define $\widehat{\lambda}(\tau, \alpha, Q_t)$ as the unique solution to the equation $\widehat{\rho}_t = (2-Q_l)\widehat{\lambda}_t(1 - \widehat{\lambda}_t)^{-1}$.Q.E.D.

Proof of Proposition 6:

If $\nu^{-1}d(R_t^n - R^s) > \lambda^*(\alpha, Q_t)$ for some t, and the government does not impose direct control over λ_t. Then, for any $\lambda_p \geq 0$, (15) implies that $\lambda_t^b > \lambda^*(\alpha, Q_t)$. Since λ_t^b is increasing in t and

$\lambda^*(\alpha, Q_t)$ is decreasing in t, we have

$$\lambda_s^b - \lambda^*(\alpha, Q_s) > \lambda_t^b - \lambda^*(\alpha, Q_t) > 0,$$

for any $s \geq t$. From (14), we have $\lim_{s\to\infty}(\lambda_s - \lambda_s^b) = 0$. This along with the inequality above implies that there exist a $t' \geq t$ such that $\lambda_{t'} > \lambda^*(\alpha, Q_{t'})$, which is a violation of the equilibrium condition that $\lambda_t < \lambda^*(\alpha, Q_t)$ for all t.Q.E.D.

Proof of Proposition 7:

For $Q = Q_l$ or $Q_l - \eta$, define $\theta(Q_t, Q)$ as follows:

$$\theta(Q_t, Q) = \frac{[QA_g + (1-Q)A_b + \gamma Q(A_g - R)] - \gamma(\gamma-1)^{-1}(2-Q)\overline{\tau}^{-1}}{[QA_g + (1-Q)A_b + \gamma Q(A_g - R)] + (2-Q)[\alpha\gamma(A_g - R) - A_g]Q_t}.$$

Apparently, for $\alpha > A_g(A_g - R)^{-1}\gamma^{-1}$ and $\bar{\tau} > \underline{\tau}$, $0 < \theta(t, Q) < \lambda^*(\alpha, t)$ and $\theta(Q_t, Q)$ is decreasing in Q_t and increasing in Q. From (9), we know that $p_{t+1}/p_t \leq \bar{\tau}$ if and only if $\lambda_t \leq \theta(Q_t, Q)$. Q.E.D.

Proof of Proposition 8:

To prove this proposition, we first show that the central bank's optimization problem can be transformed into a dynamic programing problem.

Let $\sigma_t = v^{-1}d(R_t^n - R^s)$, $h(\lambda_{t-1}, \sigma_t) = \min\{1, \nu\lambda_p + (1-\nu)\lambda_{t-1} + v\sigma_t\}$ and

$$F_t(\lambda_t, Q_l) = log\left\{\frac{\gamma - 1}{\gamma}\frac{A_g Q_t \rho_t + [Q_l A_g + (1-Q_l)A_b]}{\rho_t + 2 - Q_l}\right\},$$

where $\rho_t = (2-Q_l)\lambda_t(1-\lambda_t)^{-1}$. From (10), we have $log(Y_{t+1}) = F_t(\lambda_t, Q_l) + log(Y_t)$. Given the initial values of Y_0 and λ_{-1}, the government's optimization problem is

$$J_0(\lambda_{-1}, Y_0) = \sup_{\{\lambda_t, \bar{y}_{t+1}\}_{t\geq 0}} \sum_{t=0}^{+\infty} \beta^t log(Y_{t+1}) \tag{1}$$

subject to

$$0 \leq \lambda_t = h(\lambda_{t-1}, \sigma_t) \leq \theta(t, Q_l) \tag{2}$$

$$log(Y_{t+1}) = F_t(\lambda_t, Q_l) + log(Y_t) \tag{3}$$

or

$$0 \leq \lambda_t \leq \min\{h(\lambda_{t-1}, \sigma_t), \theta(t, Q_l - \eta)\} \tag{4}$$

$$log(Y_{t+1}) = F_t(\lambda_t, Q_l - \eta) + log(Y_t) \tag{5}$$

The problem can also be written as a dynamic programing problem

$$J_t(\lambda_{t-1}, Y_t) = \sup_{\lambda_t, \bar{y}_{t+1}} \{log(Y_{t+1}) + \beta J_{t+1}(\lambda_t, Y_{t+1})\} \tag{6}$$

subject to constraints (2) through (5). For the log utility function, the two state variable problem (6) can also be reduced to a one state variable problem. Let

$$V_t(\lambda_{t-1}) = J_t(\lambda_{t-1}, Y_t) - \frac{1}{1-\beta} log(Y_t)$$

Then, we have

Lemma 1 $\{V_t(.)\}_{t\geq 0}$ *is the unique left-continuous and bounded solution to the following functional equations:*

$$V_t(\lambda_{t-1}) = \max\{V_t^+(h(\lambda_{t-1}, \sigma_t)), V_t^-(h(\lambda_{t-1}, \sigma_t))\} \tag{7}$$

$$V_t^+(\lambda) = \sup_{0 \leq \lambda' \leq \min\{\lambda, \theta(t, Q_l - \eta)\}} \{F_t(\lambda', Q_l - \eta) + \beta V_{t+1}(\lambda')\} \tag{8}$$

$$V_t^-(\lambda) = [F_t(\lambda, Q_l) + \beta V_{t+1}(\lambda)] \mathcal{X}_{\lambda \leq \theta(t, Q_l)} \tag{9}$$

where $\mathcal{X}$ *is the indicator function. Furthermore,* V_t^+ *is bounded, continous and nondecreasing and* V_t^- *is bounded and left-continuous.*

Proof: The existence, uniqueness and continuity of the value functions follow from standard dynamic programing argument. To see that $V_t^+(.)$ is nondecreasing, note that if $x_1 \leq x_2$, then $[0, \min\{x_1, \theta(t, Q_l - \eta)\}] \subset [0, \min\{x_2, \theta(t, Q_l - \eta)\}]$, and therefore $V_t^+(x_1) \leq V_t^+(x_2)$. Q.E.D.

We now prove Proposition 8. The fact that the optimal policy will always involve direct credit control after any finite period can be proved by using the similar argument as the one used in proving Proposition 6. We now show that, for any $t \geq 0$, it is not optimal to impose direct credit control in *every* period $t+s$ for $s \geq 0$. Here, we will simply refer imposing direct credit control as control. We prove this by successively proving several claims.

Claim 1. For any $t \geq 0$, if the optimal policy in period $t+1$ is to control, then either $h(\lambda_{t-1}, \sigma_t) > \theta(t, Q_l)$ and it is optimal to control and set $\lambda_t = \theta(t, Q_l - \eta)$; or $\lambda_t = h(\lambda_{t-1}, \sigma_t) \leq \theta(t, Q_l)$ and it is optimal to have no control in period t.

<u>Proof</u>: Suppose that $h(\lambda_{t-1}, \sigma_t) \leq \theta(t, Q_l)$ and it is optimal to control in period t, then $\lambda_t < \min\{h(\lambda_{t-1}, \sigma_t), \theta(t, Q_l - \eta)\}$. Since by assumption it is optimal to control in period $t+1$, we have $V_{t+1}(\lambda_t) = V_{t+1}^{+}(\lambda_t)$. Because V_{t+1}^{+} is nondecreasing, and by the definition of λ_t, we have

$$
\begin{aligned}
& V_t^{+}(h(\lambda_{t-1}, \sigma_t)) = F_t(\lambda_t, Q_l - \eta) + \beta V_{t+1}(\lambda_t) \\
= \;& F_t(\lambda_t, Q_l - \eta) + \beta V_{t+1}^{+}(\lambda_t) \\
< \;& F_t(\min\{h(\lambda_{t-1}, \sigma_t), \theta(t, Q_l - \eta)\}, Q_l - \eta) + \beta V_{t+1}^{+}(\lambda_t) \\
\leq \;& F_t(\min\{h(\lambda_{t-1}, \sigma_t), \theta(t, Q_l - \eta)\}, Q_l - \eta) \\
& + \beta V_{t+1}^{+}(\min\{h(\lambda_{t-1}, \sigma_t), \theta(t, Q_l - \eta)\})
\end{aligned}
$$

which contradicts the definition of V_t^{+}. So if $h(\lambda_{t-1}, \sigma_t) \leq \theta(t, Q_l)$ then it is optimal not to control in period t and, therefore, $\lambda_t = h(\lambda_{t-1}, \sigma_t)$. If $h(\lambda_{t-1}, \sigma_t) > \theta(t, Q_l)$, then direct credit control is required and $\lambda_t \leq \theta(t, Q_l - \eta)$. Using the same argument as the one we have just used above, we can show that the optimal λ_t is simply $\theta(t, Q_l - \eta)$.Q.E.D.

Claim 2. If it is optimal to control in every period $t+s$ for $s \geq 0$, then it must be true that $h(\lambda_{t+s-1}, \sigma_{t+s}) > \theta(t+s, Q_l)$ and $\lambda_{t+s} = \theta(t+s, Q_l - \eta)$ for every $s \geq 0$.

<u>Proof</u>: If for some $s \geq 0$ we have $h(\lambda_{t+s-1}, \sigma_{t+s}) \leq \theta(t+s, Q_l)$, then, by the assumption that it is optimal to control in $t+s+1$ and by Claim 1 we know that it is not optimal to control in period $t+s$, which is contradictory to our assumption. Therefore, the inequality has to hold for every $s \geq 0$. From Claim 1, then, we know that $\lambda_{t+s} = \theta(t+s, Q_l - \eta)$ for every $s \geq 0$.Q.E.D.

Claim 3. If Assumption 4 holds, then it can not be true that there exist a $t \geq 0$ such that, for every $s \geq 0$, $h(\lambda_{t+s-1}, \sigma_{t+s}) > \theta(t+s, Q_l)$ and $\lambda_{t+s} = \theta(t+s, Q_l - \eta)$.

<u>Proof</u>: Assume that for some $t \geq 0$, $h(\lambda_{t+s-1}, \sigma_{t+s}) > \theta(t+s, Q_l)$ and $\lambda_{t+s} = \theta(t+s, Q_l - \eta)$

for every $s \geq 0$. Then, letting $s \to +\infty$ and taking limits we have $\lim_{s\to+\infty} \lambda_{t+s} = \bar{\theta}(Q_l - \eta)$ and

$$h(\overline{\theta}(Q_l - \eta), \overline{\sigma}) \geq \overline{\theta}(Q_l)$$

By the definition of the function $h(.,.)$ and the fact that $\lambda_p \leq \overline{\theta}(Q_l)$, we have

$$h(\bar{\theta}(Q_l - \eta), \overline{\sigma}) \leq v\overline{\theta}(Q_l) + (1 - v)\bar{\theta}(Q_l - \eta) + v\overline{\sigma}$$

So the above two inequalities imply

$$\bar{\theta}(Q_l) - \bar{\theta}(Q_l - \eta) \leq v(1 - v)^{-1}\overline{\sigma}$$

which is contradictory to Assumption 4.Q.E.D.

Thus, Proposition 8 has been proved.